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An analysis of the dynamic behaviour of earnings distributions

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An analysis of the dynamic behaviour of earnings distributions is conducted here in three ways. First, the method of dimensional analysis, in the context of Buckingham's Π theorem, is employed to demonstrate that earnings distributions, which are almost always dynamic in character, should, under certain conditions and a special coordinate transformation, be self-similar and time-invariant. Application of the theorem to some empirical data, pertaining to Canadian income, fully confirms this finding.

Second, an economics-based model, incorporating the concept of a shock-free economy in competitive equilibrium, is developed to provide an intuitive account of the conclusions reached above. Testing the model's predictions against data once again yields satisfactory agreement between the two. The model is finally extended to account for labour-force mobility across income brackets. The outcome of this, when compared with empirical data, could reflect the degree of homogeneity, and even discrimination, in a labour force.

I. INTRODUCTION

The distribution of labour earnings (hereafter also referred to as income or wages) of a labour force is an important economic indicator, which is useful for providing a measure of welfare in terms of social (in)equality. Within the economy as a whole, the distribution is always observed to vary with time, with the nominal wage rate generally increasing over time for all workers. The dynamics of this evolution appear to be influenced by various factors, which include inflation and interest rates, as well as by technological progress and other market conditions, in general. These dynamic effects are not necessarily synonymous with distributional effects, which shift the distribution of the resources to benefit one part of the society and, at the same time, harm the other part (Stonier and Hague, 1967; Lambert, 1993). Such shifts, though, are customarily neglected in macroeconomics, where aggregates are analysed (Barro and Lucas, 1994).

Theoretical and empirical investigations on the time-dependent movements of income distributions abound in the literature. Many of these emphasize the importance of the mean and distribution of incomes in describing relative welfare (Sen, 1973; Iritani and Kuga, 1983), whereas others depend on econometric methods to assess changes in inequality over time. In all, the majority of the empirical studies conducted on US earnings data provide evidence that, at least within the past few decades, inequality has been on the rise (Bishop *et al.*, 1991, and references therein).

The dynamic behaviour of income distributions *across* countries has also been looked at, but this was carried out mainly in the context of convergence and divergence of economic growth (Quah, 1996; Barro and Sala-i-Martin, 1995, and references therein). To date, numerous relevant studies have appeared and, even though some of them are in conflict, they do provide useful measures of the current state of income dynamics across countries, and, in addition, attempt to extrapolate the future from present and past data. These studies have generated a wide array of models, leading to, among others, the notions of β- and σ-convergence. We shall return to these later.

Within the scope of intersectoral earnings, which is most relevant to this work, the answer to whether or not the observed time-dependent shifts in earnings distributions signal the redistribution and/or growth of wealth is not so straightforward. For instance, a positive shift in the earnings distribution does not necessarily mean redistribution of real wealth, or a corresponding increase in it. For this reason, therefore, different possibilities for probing into this matter

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have been proposed, one being to search for dynamic variations in the Lorenz curve (Atkinson, 1976). Should these variations be present there, redistribution of real wealth is then taking place.

To help further explain the nature of these time-dependent movements in the earnings distributions, many theoretical, as well as semi-empirical, works in economics have been advanced over the years. The emphasis of many of these has revolved primarily around bringing together all, or parts, of the contributing economic factors in the form of a model, with the intention of better understanding how and why the earnings distribution behaves the way it does, and even, it was hoped, to predict its time-wise progression (Kurz, 1979; Musgrave and Musgrave, 1989).

Here, as well, we attempt to examine and explain the dynamic changes in the earnings distributions, but through methodologies that are different from those commonly found in the economics literature. The methods used here incorporate, for the most part, well-known engineering concepts that have proved very beneficial in data analysis and modelling of physical phenomena. Very briefly, these methods, which are described in full in Sections II to IV, are as follows:

(i) The first approach, which is outlined in Section II, is quite unlike the standard ones found in the economics literature. The notion itself, which is purely mathematical, is borrowed from ideas that have long existed in the engineering literature and which have so far proved very effective and popular. It is known as 'dimensional analysis', or, more formally, as Buckingham's Π theorem (Buckingham, 1914).

The method of dimensional analysis, notwithstanding, is not new to economics (Jong, 1967), but, for some reason, it has not caught on very widely. We aim here to demonstrate its simplicity, usefulness and effectiveness through an example that involves the dynamics of earnings distributions, concentrating primarily on the set of data presented in Section II. We should mention that these data have been chosen arbitrarily, purely for illustrative purposes. Thus the approach is by no means restricted to only these, nor to their respective time frames.

(ii) The second method, explained in Section III, is more in line with standard economic modelling, as it attempts to describe how the earnings distribution should evolve in time under competitive equilibrium and in the absence of economic shocks. In this the model also justifies, as well as provides, the economic meaning and intuition behind the results obtained in (i) above. The methodology, nevertheless, concentrates not on earnings across different countries or sectors, but on earnings within sectors of populations. Moreover, it differs from previous works related to welfare studies in that it makes no specific reference to inequality measures, not does it seek to predict the future from current or past data.

As we proceed, we shall demonstrate through basic principles how any given earnings distribution should evolve in a shock-free economy (to be defined later), which is in competitive equilibrium. Initially, a simple model of this type of economy is constructed, from which a governing partial differential equation that describes only the dynamic evolution (and *not* the shape) of the earnings distribution, is deduced. The solution satisfying this partial differential equation is *generic* in form, and possesses several interesting properties. This generic solution is later compared to empirical data, and the resulting implications are discussed in detail.

In deriving the model, we shall introduce a new notion of convergence, which is analogous in idea to, but fundamentally different from, the above-mentioned β- and σ-convergence. This version seems to apply here to the time-wise progression of the earnings distributions *within* various populations of sectors of populations (white and non-white families, all individuals, unattached individuals, men, women, etc.) in the USA, Canada and Australia, for which extensive data are available, and proves that under a certain coordinate transformation, which coincides exactly with that generated by the Π theorem (as discussed in (i) above), these distributions have not changed significantly over many years, and even several decades in some situations. In other words, the earnings distributions considered here may already have converged to some kind of a steady state. Earnings across sectors, however, were not investigated here.

(iii) Finally, in Section IV, the model is extended further to demonstrate, first, how free cross-wage mobility could affect the distribution of earnings in a labour force, and second, that self-similarity and time-invariance are once again satisfied. Once these are accomplished, an elementary statistical analysis is used to evaluate the spread of the available data around this simple, theoretical model

At this point, it is important to emphasize that none of the methodologies adopted here is econometric in nature. That is to say, we make no attempt whatsoever either to produce regression equations or compute regression coefficients; nor do we intend to compare a restricted model with an unrestricted one. Carrying out such an analysis here is not only beyond the scope of this work, but it would also divert the attention from the main theme. Our approach, to be revealed shortly, employs mostly engineering principles to demonstrate how a given earnings distribution should, in general, evolve in time under certain imposed assumptions.

II. DIMENSIONAL ANALYSIS AND ITS APPLICATION TO THE DYNAMICS OF EARNINGS DISTRIBUTIONS

Preliminary analysis

As noted earlier, the method of dimensional analysis, in the framework of Buckingham's Π theorem, is applied widely in engineering practices, especially for compressing empirical data to more manageable formats. Evidently, however, the theorem has, to date, had little impact in economics, which perhaps explains why most economists are unaware of it. Even though some economics-related works discussing its virtues do exist (see, e.g., Jong, 1967), they are, unfortunately, few and scattered.

Our intention here is to use the theorem to assess the dynamics of earnings distributions. Since applications of it in economics are so rare, we shall presume that the reader is not familiar with the subject. Therefore, it is in some instances necessary to explain some of its insights and objectives. Nevertheless, our explanations will be concise, bearing in mind that detailed descriptions would only be repetitions of what is already available elsewhere. We should, however, mention that the engineering literature covering this subject is voluminous, so the interested reader may consult any relevant, practical text (e.g. Eskinazi, 1965).

We begin with the main parameter of the problem, which is the earnings distribution density function, $p(w, t)$. This is defined by

$$p(w, t) \equiv \frac{1}{N(t)} \lim_{\Delta w \to 0} \frac{\Delta N(w, t)}{\Delta w} \tag{1}$$

where t is time, w is the wage or earnings rate, $N(t)$ is the total labour force and $\Delta N(w, t)$ is the portion of the labour force earning w and falling within wage bracket Δw. Obviously, therefore, $p(w, t)$ must satisfy the normality condition

$$\int_0^\infty p(w, t) \mathrm{d}w = 1 \qquad \forall t \geqslant 0 \tag{2}$$

Since all the variables in Equation 1 are generally estimable from histograms or other types of data, calculation of $p(w, t)$, given such data, is straightforward using the approximation

$$p(w, t) \approx \frac{1}{N(t)} \frac{\Delta N(w, t)}{\Delta w} \tag{3}$$

Next, we propose a general functional form, such as the following:

$$p(w, t) = p(w, r(t), i(t), \Psi(t), \mathrm{M}(t), \Lambda(t), \ldots) \tag{4}$$

to describe the indirect influence of time, if any, on p through the various economic indicators. Here, $r(t)$ and $i(t)$, respectively, are the interest and inflation rates, while $\Psi(t)$ and $\mathrm{M}(t)$, respectively, represent technology and money supply. Finally, $\Lambda(t)$ depicts other observable and non-observable characteristics of the economy that could influence the dynamics of p. These, however, are numerous and so we shall not try to delve into them here. Instead, let us suppose, for simplicity, that the *total* number of these effective variables or indicators is n.

It is important now to observe that some of the above-mentioned indicators – within the set $\{r(t), i(t), \Psi(t), \mathrm{M}(t), \Lambda(t), \ldots\}$ – might be related to each other. For instance, it is recognized that, among other factors, the interest rate is linked to inflation rate, and the inflation rate to the money supply.

Hence, to avoid related, but unwarranted, complications, let us assume that these interrelationships are known, so that, by appropriately eliminating and combining certain parameters within the original group, we can condense the set to consist of only those that are entirely independent of each other. In other words, we reduce the set $\{r(t), i(t), \Psi(t), \mathrm{M}(t), \Lambda(t), \ldots\}$ to $\{x_1(t), \ldots, x_m(t)\}$, with $m \leqslant n$, where each $x_i(t)$, belonging in some respect to the set $\{r(t), i(t), \Psi(t), \mathrm{M}(t), \Lambda(t), \ldots\}$, is *independent* of all other x_js, for $j \neq i$. This, then, allows us to re-express Equation 4 as

$$p(w, t) = p(w, x_1(t), \ldots, x_m(t)) \tag{5}$$

which indicates that, aside from w, there are m such potentially time-dependent, but *mutually independent*, economic factors – i.e. the set $\{x_1(t), \ldots, x_m(t)\}$ – that affect $p(w, t)$. Obviously, Equations 4 and 5 are identical, except that (4) represents $p(w, t)$ in terms of the economic indicators as we know them, while (5) reconstructs $p(w, t)$ as a function of the mutually independent indicators, whatever they may be. Since the object of this part of the work is purely mathematical, we shall make no attempt of any kind to extract the mutually independent parameters, $\{x_1(t), \ldots, x_m(t)\}$, from the group $\{r(t), i(t), \Psi(t), \mathrm{M}(t), \Lambda(t), \ldots\}$ and, instead, proceed directly with the analysis.

So, with p given in (5), we write the jth moment of the distribution as

$$\overline{w^j}(x_1, \ldots, x_m) \equiv \int_0^\infty w^j p(w, x_1, \ldots, x_m) \, dw, \qquad j \geqslant 1 \tag{6}$$

where, for brevity, the time-dependence sign, (t), has been omitted. Obviously, the 0th moment is unity by means of Equation 2, and $\overline{w^1}$ is the average wage, which, for convenience, is being denoted here by $\bar{w}$. Also, it is useful to note that, given the data at any time, each of these moments is computable and representable by a mere number.

We digress here for a moment to comment on the value of m. According to the above, m could be any integer greater than or equal to zero. This, obviously, poses the question, what is m? Here, as in any other sensible way of modelling a difficult situation, we shall rely on simplifying assumptions, one being that m is finite and small, perhaps on the order of 2 or, at most, 5. This merely means that only a few *mutually independent* economic parameters are needed to capture most of the dynamic variations in the income distribution. Evidence of this lies in the more common semi-empirical curve fits that successfully represent some wage distributions – fits that are well characterized by only two parameters, such as the mean and the standard deviation, and, maybe, in addition to one or two more (in some instances, for example, the tail portion is also an important characteristic). In light of this, m is 'small' and, most certainly, it does not tend to infinity.

Returning to the problem, we define m such moments by $\{\overline{w^1} = \bar{w}, \overline{w^2}, \ldots, \overline{w^m}\}$, all of which may be time-dependent. Obviously, this involves the additional assumption that all these moments exist. Although the generality of this might be questioned, our previous contention that m is small, plus the fact that actual income distributions are limited in range, serve only to reinforce it.

Now, since the probability density function, p, depends on w and the mutually independent parameters $\{x_1, \ldots, x_m\}$, then, by virtue of Equation 6, each of these moments is, in turn, a function of $\{x_1, \ldots, x_m\}$. This, therefore, allows us to write:

$$\overline{w^1} = \overline{w^1}(x_1, \ldots, x_m) = \bar{w} \quad (7.1)$$

$$\overline{w^2} = \overline{w^2}(x_1, \ldots, x_m) \quad (7.2)$$

$$\vdots \qquad\qquad \vdots$$

$$\overline{w^m} = \overline{w^m}(x_1, \ldots, x_m) \quad (7.m)$$

Because the above represent m equations of the m moments as functions of $\{x_1, \ldots, x_m\}$, then each of the x_is should, in turn, be recoverable in terms of $\{\bar{w}, \overline{w^2}, \ldots, \overline{w^m}\}$ – that is

$$x_1 = x_1(\bar{w}, \overline{w^2}, \ldots, \overline{w^m}) \quad (8.1)$$

$$x_2 = x_2(\bar{w}, \overline{w^2}, \ldots, \overline{w^m}) \quad (8.2)$$

$$\vdots \qquad\qquad \vdots$$

$$x_m = x_m(\bar{w}, \overline{w^2}, \ldots, \overline{w^m}) \quad (8.m)$$

Next, we substitute Equations 8.1 through (8.m) into (5) to obtain a functional expression for the probability density in terms of w and all the moments, i.e.

$$p(w, t) = p(w, \bar{w}(t), \overline{w^2}(t), \ldots, \overline{w^m}(t)) \quad (9)$$

What we have accomplished so far by converting Equation 5 to 9 is an expression for the probability density function, p, in terms of w and the moments, all of which are computable quantities given the earnings distribution data and Equation 6. This form is preferred over its counterpart, Equation 5, because the latter entails difficulties is precisely defining the parameters x_i, $i = 1, \ldots, m$. Nevertheless, Equation 9 contains all the information inherent in 5.

Application of dimensional analysis in the framework of Buckingham's Π theorem

Reducing the probability density function, p, from Equation 5 to the more tractable Equation 9 enables us now to apply the method of dimensional analysis in the following manner. First, we note from Equation 1 that p acquires the 'dimension' or 'fundamental unit' of inverse wage rate, with wage rate being given in (currency/time). Denoting this dimensional unit by Θ, i.e.

$$\langle \Theta \rangle = \text{currency/time} \quad (10)$$

where the notation $\langle * \rangle$ is used here to denote 'the dimensional unit of *,' leads to

$$\langle p \rangle = \Theta^{-1} \quad (11.1)$$

$$\langle w \rangle = \Theta \quad (11.2)$$

$$\langle \bar{w} \rangle = \Theta \quad (11.3)$$

$$\langle \overline{w^2} \rangle = \Theta^2 \quad (11.4)$$

$$\vdots \qquad\qquad \vdots$$

$$\langle \overline{w^m} \rangle = \Theta^m \quad (11.m+2)$$

Thus, based on the above, there is a total $m + 2$ 'dimensional' variables – i.e. p, w, $\bar{w}$, $\overline{w^2}$, …, $\overline{w^m}$ – each characterized, in one way or another, by the *single* fundamental unit Θ.

Now, allowing for notational consistency among the indices, Buckingham's theorem states that, given a physical equation:

$$F(z_1, z_2, z_3, \ldots, z_{m+2}) = 0 \quad (12)$$

where z_1, z_2, z_3, …, z_{m+2} are the dimensional variables pertinent to the problem that Equation 12 describes, there are $(m + 2 - k)$ dimensionless Π variables that describe the same problem as

$$F(z_1, z_2, z_3, \ldots, z_{m+2}) = \Phi(\Pi_1, \Pi_2, \Pi_3, \ldots, \Pi_{m+2-k}) = 0 \quad (13)$$

With k being the number of fundamental units in the problem, then, according to the theorem, since $k = 1$ in our problem (i.e., there is only one, which is Θ), there should be only one 'basis' parameter with respect to which all the other variables are to be 'non-dimensionalized'. This, subsequently, results in $m + 2 - k = m + 1$ *dimensionless* Π *variables.* Recognizing that the choice for this basis parameter is arbitrary, we choose it here, solely for convenience, to be $\bar{w}$, recalling that its dimension its Θ, as given in (11.3).

With $\bar{w}$ as the basis parameter, therefore, we non-dimensionalize the remaining variables – p, w, $\overline{w^2}$, …, and $\overline{w^m}$ – relative to it. This could be done here easily because there is only one basis parameter. Otherwise, when more than one is involved, the theorem provides a simple technique for non-dimensionalization.

Thus, based on $\bar{w}$, the set of $m + 1$ *dimensionless variables* in the present problem is $\{\bar{w}p, w/\bar{w}, \overline{w^2}/\bar{w}^2, \ldots, \overline{w^m}/\bar{w}^m\}$, which now, in compliance with the theorem, enables us to re-express the dimensional Equation 9 in the following dimensionless functional form:

$$\bar{w}p = f(w/\bar{w}, \overline{w^2}/\bar{w}^2, \ldots, \overline{w^m}/\bar{w}^m) \quad (14)$$

Without loss of generality, we multiply both sides of (14) by the ratio $w/\bar{w}$ to obtain

$$wp = g(w/\bar{w}, \overline{w^2}/\bar{w}^2, \ldots, \overline{w^m}/\bar{w}^m) \tag{15}$$

which is another, but equivalent, version of (14). The latter is probably more convenient only because the dependent variable, which is the product wp, is indicative of the share of total earnings, a more common and practical economic indicator than the product $\bar{w}p$.

On returning to Equation 15, it is apparent that, aside from the fact that it embodies all the information that its equivalent, Equation 5, does, it has certain advantages over it. First Equation 15 is non-dimensional, meaning that it is independent of whether the wage rate is expressed in dollars per week or pounds per year. This way, different data sets belonging to different time scales and currencies could be compared against each other in a more straightforward fashion. Second, the number of *independent* variables in Equation 15 is less than that in (5) (i.e., m in Equation 15 as opposed to $m + 1$ in Equation 5). This, by itself, is an improvement since it helps to facilitate data analysis.

More important, however, is Equation 15's implication that the quantity, wp, should depend on the ratio $w/\bar{w}$, as well as on the moment ratios $\overline{w^2}/\bar{w}^2, \ldots,$ and $\overline{w^m}/\bar{w}^m$, through which the effects of time should enter. Thus, if these moment ratios were constant, or even approximately constant, over time, one could infer that a simple coordinate transformation on a given p versus w data set – i.e. replotting it as wp versus $w/\bar{w}$ instead – should cause the distribution of shares at different time periods to converge on to a single, time-invariant curve, depending *only* on $w/\bar{w}$. This will be illustrated more clearly through the example that follows.

Data analysis

To further elucidate the points made above, we have selected to work with an actual data set of income distributions, belonging to 'unattached individuals' in Canada. As mentioned in Section I, this data set was chosen purely arbitrarily, and the analysis that follows could be applied as easily to other data sets.

These data, which come from *Statistics Canada* (1994), were histogram-type originally, tabulated as the fraction of individuals, $\Delta N(w, t)/N(t)$, falling within income bracket Δw. For convenience, we have rearranged these in the form of probability density functions, $p(w, t)$, using Equation 3, and plotted them against the yearly wages, w, in current Canadian dollars. The outcome of this is displayed in Fig. 1. The only reason for using nominal values instead of real was to avoid bringing in an additional indicator, which is the price level. Otherwise, whether one implements real or

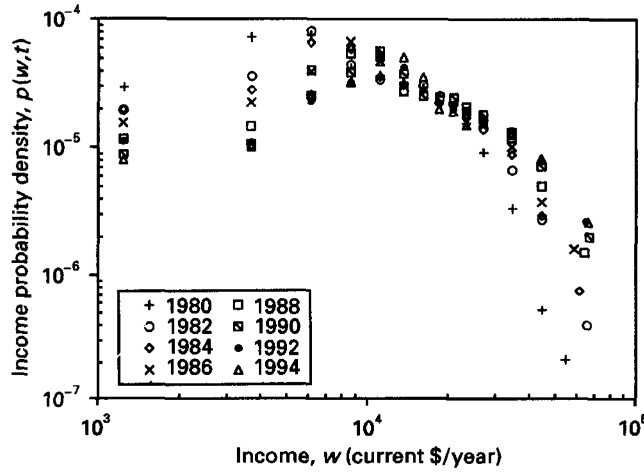

Fig. 1. *The income distribution density function of 'unattached individuals' in Canada* (Statistics Canada, *1994*) *plotted as* p(w, t) *versus* w. *The data cover the time period between 1980 and 1994, during which the average income increased from 11 574 to 23 746 in current Canadian dollars and the estimated numbers of 2.653E6 to 3.836E6*

nominal terms for the wages makes no difference whatsoever in the final outcome of the analysis.

Overall, it is obvious from Fig. 1 that over the time frame covered, which is from 1980 to 1994, there appears to be considerable scatter, especially taking into account the logarithmic nature of the scales. The changes in p that take place throughout these years are, according to Equation 5, indicative of the shifts in the different economic indicators, such as inflation, interest rates, etc., all rooted within the set $\{x_1, \ldots, x_m\}$, whatever m happens to be. Thus, the evolution of the nominal earnings distribution in Fig. 1 is a manifestation of all these effects over the time period considered.

The moment ratios for this data set were then computed to see how they vary with time. The result of this, carried out up to 5th order, is presented in Fig. 2. For notational simplicity, we have defined a_j to be the jth moment ratio, that is:

$$a_j \equiv \frac{\overline{w^j}}{\bar{w}^j} \tag{16}$$

Curiously, the behaviour of these moment ratios appears to be fairly constant over the time interval tested. This could indicate that there were no major shocks or changes affecting the earnings distribution during this time period (except, perhaps, in 1982, where a relatively weak upheaval is observed in the higher moments). As a result of this quasi-constant behaviour, the product wp should, according to Equation 15, be a function of $w/\bar{w}$ only, remaining more or less *independent of time*. Simply stated, a plot of wp versus $w/\bar{w}$ for this data set should cause the data points, which are quite scattered in the p–w–t space (see Fig. 1), to congregate

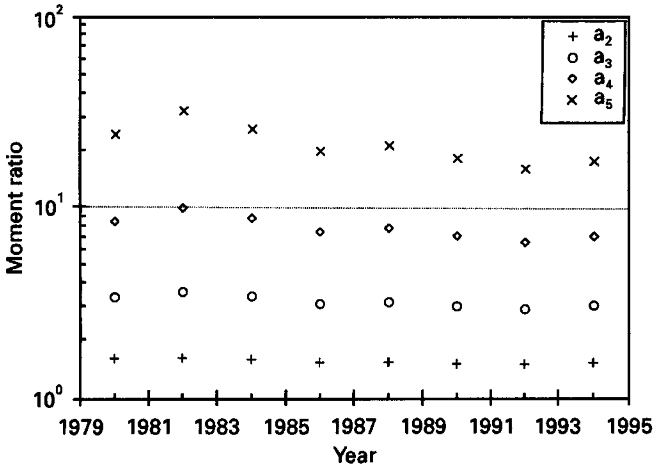

Fig. 2. *Time dependence of the moment ratios, up to the 5th moment. Here,* a_j *represents the moment ratio* $\overline{w^j}/\bar{w}^j$*, as defined in Equation 16. Note that all the ratios are fairly constant over the time period considered*

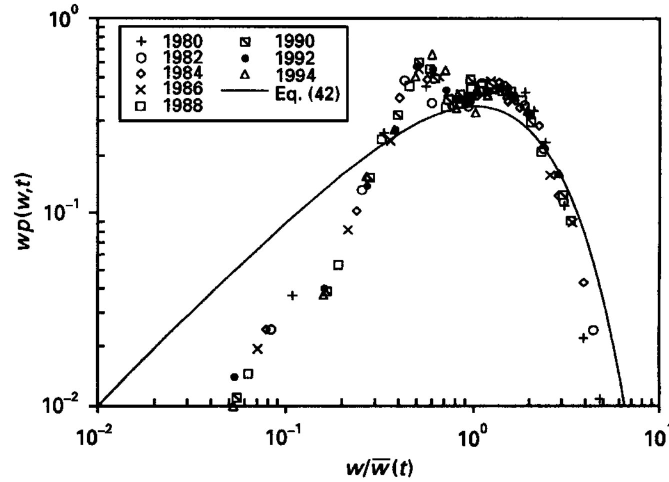

Fig. 3. *The income distribution of Fig. 1 plotted as* wp *versus* w/w̄. *Note the convergence of the data points under this simple coordinate transformation*

along a single curve, exhibiting, if any, only minute signs of time dependence.

Interestingly enough, in Fig. 3, where *wp* is plotted against $w/\bar{w}$, this behaviour is clearly demonstrated, attesting, therefore, to the success of the method, as well as to the general applicability of Equation 15. The non-dimensionality that is imposed here corresponds with the removal of the effects of the monetary unit (i.e. inflation) and, thus, helps to portray the *real* nature of earnings. The solid line, on the other hand, which is present in this figure, is the result of theoretical analysis that will be discussed in Section IV.

What the convergence of the data in Fig. 3 means is that, although Fig. 1 displays considerable time-dependent variations in earnings between 1980 and 1994, there has been very little change in the quantity, *wp*, if one were to view it from the perspective of Fig. 3. This clearly suggests that redistribution of real wealth is almost absent from this data set.

Summary and concluding remarks

The method of dimensional analysis, within the context of Buckingham's Π theorem, has been used here to prove that, under special circumstances and a certain coordinate transformation, the distribution of wages should display self-similar and time-invariant properties. As demonstrated, the method is almost entirely mathematical and, thus, it is neither capable of theoretically explaining the observed economic trends, nor could it forecast the shape of the distribution curves. Instead, through systematic data reduction, it is able to unfold functional characteristics that would otherwise remain hidden. The main appeal of this theorem, therefore, lies in its simplicity and practicality, especially in reformulating data to the point where intrinsic trends and patterns, if any, should begin to emerge.

In addition to the conclusions discussed earlier, we find that, in the absence of shocks or other major changes in the economy (whose presence should be visible through the transient behaviour of the moment ratios), the progress of the earnings distribution, in its entirety, is characterizable by a single, time-dependent element, namely the mean wage. This is because $wp(w, t)$ depends only on the ratio $w/\bar{w}$, whereas, at the same time, all higher moments depend on $\bar{w}(t)$. This further indicates that the dynamic effects of the economy on the income distribution, however intertwined and complex they may be, all coalesce to form this individual entity, which is $\bar{w}(t)$. This is interesting since, in such cases, the standard deviation is automatically eliminated as an additional independent characteristic of the distribution.

We should mention in passing that behind the coordinate transformation implicit in Equation 15 there is a theoretical explanation that could be derived rigorously using economics-based arguments, as shown next in Section III. These describe the time-wise progression of the earnings distribution in a shock-free economy under competitive equilibrium. Section III also presents several more of these examples, all in full agreement with the conclusions arrived at here.

III. THE DYNAMIC EVOLUTION OF THE EARNINGS DISTRIBUTION UNDER COMPETITIVE EQUILIBRIUM

Here we shall demonstrate how a typical distribution of earnings, belonging to a sector of a population, should evolve in time in an economic setting that is in competitive

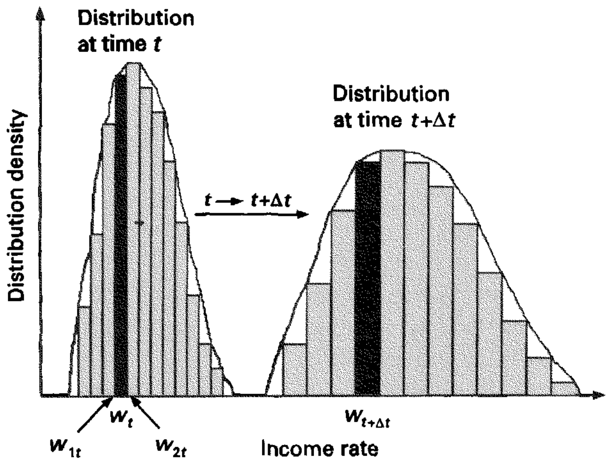

Fig. 4. *Example of the dynamic behaviour of the income distribution function. Focus is on one of the income intervals,* w_t, *which is highlighted, as it evolves in time*

equilibrium and free of shocks. First, we introduce the relevant notations, definitions and assumptions, and then develop and analyse a mathematical model of such an economy. Finally, we assess the predictions of this model and compare them with empirical data.

Notations, definitions and assumptions

A schematic example of how an earnings distribution could evolve in time is illustrated in Fig. 4. Consider first the distribution at time t, which is broken into narrow wage 'bands', 'intervals', or 'brackets'. Here, w_t is the mean wage rate (e.g. in dollars) in the bracket and the subscript t corresponds to the time at which the observation was made. The interval containing w_t is shown to occupy the region between w_{1t} and w_{2t}, with Δw_t being defined by

$$\Delta w_t = w_{2t} - w_{1t} \tag{17.1}$$

Also, given w_{1t} and w_{2t}, w_t may be estimated from

$$w_t \approx (w_{2t} + w_{1t})/2 \tag{17.2}$$

Such representations of wage distribution data, particularly in the form of histograms and probability density functions, is very common. Generally, the fraction of the labour force earning wages that fall within the interval w_{1t} to w_{2t} is plotted against the interval mean, w_t. Whether real or nominal wages are to be used makes no difference in the final results.

We now let every wage interval consist of several firms, each offering at any given time the same wage rate, w_t, to all its workers. Denoting the total number of workers in interval w_t by $\Delta N(w_t)$, the entire labour force, $N(t)$, which may or may not vary with time, is therefore

$$N(t) = \sum_{w_t} \Delta N(w_t) \tag{18}$$

This, of course, is the sum of the workers in all the wage intervals.

In general, the output from different firms within an interval may be different, and sold at different prices to the consumer. Furthermore, a firm that offers various levels of wages to its workers at any given time may occupy subdivisions in different intervals. This work, we should emphasize, does not delve into these generalities. The reason for this is that while they only tend to complicate the matter, they do not alter, in any way, the results derived hereafter based on our simplifications.

Returning now to the analysis, if we multiply and divide the right-hand side of Equation 18 by Δw_t and define the earnings distribution density function, $p(w_t)$, as

$$p(w_t) = \frac{1}{N(t)} \lim_{\Delta w_t \to 0} \frac{\Delta N(w_t)}{\Delta w_t} \tag{19}$$

we obtain

$$\int_0^\infty p(w_t)dw_t = 1 \tag{20}$$

which is Equation 18 in continuous form. Given $p(w_t)$, therefore, one should be able to compute the time-dependent average wage, $\bar{w}(t)$, of the distribution from

$$\bar{w}(t) = \int_0^\infty w_t p(w_t)\, dw_t \tag{21}$$

For obvious reasons, however, discrete representation of (20) and (21) are used more widely in practice.

The time-wise evolution of an earnings interval

We continue here by focusing on a single interval, Δw_t, that is centred around wage w_t, follow its progress in time and observe how certain of its properties change. Technically speaking, therefore, we are analysing the problem from a 'moving frame of reference', which is also known as 'panel data' in econometrics terminology.

Once again we refer to Fig. 4, which exemplifies the movement of the earnings distribution density function in time. Let us first concentrate on interval w_t at time t, and follow it in time till it increases to $w_{t+\Delta t}$. This increase in wage may, for instance, be due to incremental pay raises. For clarity, the segments w_t and $w_{t+\Delta t}$ have been isolated and depicted separately in Fig. 5.

We then assume that the firm offering w_t adjusts the wage of its workers according to an exogenously determined *average wage* growth rate, $\zeta(t)$, which is given by

$$\zeta(t) = \frac{d\ln(w_t)}{dt} = \frac{d\ln(\bar{w}(t))}{dt} \tag{22}$$

where $\bar{w}(t)$ is defined in (21). Simply stated, this implies that all wages within the economy are affected equally. This assumption, which is also a consequence of Gibrat's Law, is

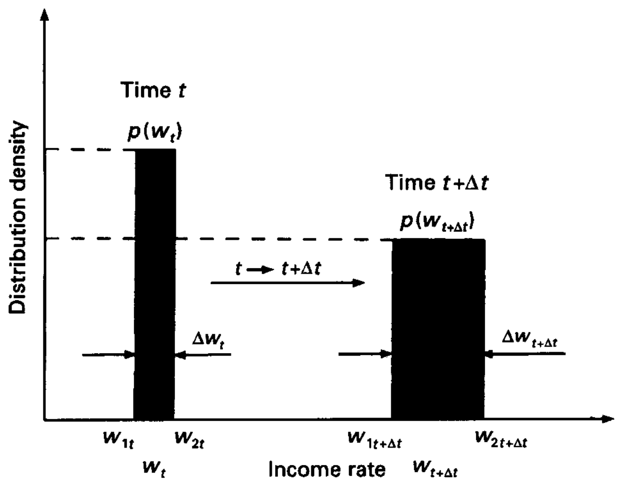

Fig. 5. *The income interval,* w_t, *highlighted in Fig. 4, is shown separately here as it changes to* $w_{t+\Delta t}$. *In the process,* $p(w_t)$ *and* Δw_t *change to* $p(w_{t+\Delta t})$ *and* $\Delta w_{t+\Delta t}$, *respectively*

reasonable because as competitive firms are forced to take prices as given, they also have to offer their employees pay raises that follow some outside, common factor; otherwise the employees would seek to work elsewhere. With this in mind, therefore, we note from Figs 4 and 5 that, on following a firm's wage development over time, w_t changes to $w_{t+\Delta t}$, and so do Δw_t, $\Delta N(w_t)/N(t)$ and $p(w_t)$ to $\Delta w_{t+\Delta t}$, $\Delta N(w_{t+\Delta t})/N(t+\Delta t)$ and $p(w_{t+\Delta t})$, respectively.

Having now set the proper terminology for the 'moving frame of reference' point of view of the wage distribution, we are in a position to develop a model for a shock-free competitive-equilibrium economy, predict from it the time-wise evolution of the earnings distribution density function, compare the results with actual data and, finally, discuss the outcome.

A model of a shock-free economy in competitive equilibrium

We begin by letting $\Delta\pi(w_t)$ be the time-dependent profit generated within wage interval Δw_t that is centred around w_t, and write it as

$$\Delta\pi(w_t) = P(w_t)\Delta f^s(w_t)N(t) - w_t\Delta N(w_t) \tag{23}$$

where $P(w_t)$ is the unit price of the commodity produced in that wage interval and $\Delta f^s(w_t)$ is the interval's output supply rate of the commodity *per capita.* Note that the first term on the right-hand side of Equation 23 is the returns to the interval, while the second denotes the cost in terms of total wages paid to the employees. Here, the interval is assumed to supply a homogeneous product to the consumer at unit price $P(w_t)$. This restriction of product homogeneity can easily be relaxed so that the variety of products having different prices may emanate from the same interval. Once again, such generalizations should not have any effect on the final results. The simplified, homogeneous-product approach is taken here only to reduce the complexities involved in the derivations.

Next, we assume that within any interval, several firms compete against each other to sell the same product. As a result, (i) the growth rate of the average wage within each interval, $d[\ln(w_t)]/dt$, is governed by a common outside force, and acquires the form of Equation 22 (that is, all wages and wage brackets in this competitive economy are impacted equally by the economy as a whole), (ii) the selling price, $P(w_t)$, charged by all firms in a given interval is the same, and (iii) the firms operate at constant-returns-to-scale technology, thereby leading to a zero profit margin. This last statement is due to the fact that every decreasing-returns-to-scale technology can be thought of as a constant-returns-to-scale technology by viewing profit as either an opportunity cost or economic rent. Based on this accounting convention, therefore, firms make zero profit at equilibrium (Varian, 1992).

Thus, after setting $\Delta\pi(w_t) = 0$ in Equation 23, rearranging it and dividing both sides by Δw_t, we obtain

$$\frac{w_t}{N(t)}\frac{\Delta N(w_t)}{\Delta w_t} = P(w_t)\frac{\Delta f^s(w_t)}{\Delta w_t} \tag{24}$$

We now apply the market-clearing condition that

$$\Delta f^s(w_t) = \Delta f^D(w_t) = \Delta f^*(w_t) \tag{25}$$

where $\Delta f^D(w_t)$ is the *per capita* demand for the output from interval w_t and $\Delta f^*(w_t)$ is the equilibrium quantity. Thereby, substituting (25) into (24) and defining $\mu^*(w_t)$ as

$$\mu^*(w_t) \equiv \lim_{\Delta w_t \to 0}\left\{P(w_t)\frac{\Delta f^*(w_t)}{\Delta w_t}\right\} \tag{26}$$

yields

$$w_t p(w_t) = \mu^*(w_t) \tag{27}$$

after utilizing the expression for $p(w_t)$ in Equation 19. Simply stated, $\mu^*(w_t)$ combines the *per capita* consumption of interval w_ts product with the wage range, Δw_t, and price, $P(w_t)$, into a single entity.

The meaning behind $\mu^*(w_t)$ becomes clearer once we examine it in aggregate. Note, for instance, that, at the aggregate level, the numerator on the right-hand side of Equation 26 scales with the average *per capita* GDP, which is nothing but the average *per capita* productivity or wage, $\bar{w}$, and the denominator with the standard deviation, σ, where σ^2 is the variance of the earnings distribution. Consequently, the quantity $\mu^*(w_t)$ is on the order of the ratio $\bar{w}/\sigma$, a quantity which may or may not vary with time.

We now define the terminology 'shock-free' within the context of earnings dynamics as a $\bar{w}/\sigma$ ratio that remains more or less constant throughout the time period of interest. This merely implies that the standard deviation, σ, of the earnings distribution changes directly as the mean income, which is to say that as the mean of the distribution increases (decreases), the distribution widens (narrows) proportionately. In contrast to the notion of σ-convergence that was mentioned earlier (Barro and Sala-i-Martin, 1995), we note that the one developed here – i.e. convergence in $\bar{w}/\sigma$ – is fundamentally different and leads to another type of steady state. Notwithstanding, the two concepts of convergence – in real σ and in the ratio $\sigma/\bar{w}$ – become one and the same if the average nominal wage, $\bar{w}$, is directly proportional to the price level.

Furthermore, on returning to the application of Buckingham's Π theorem in Section II, we notice that the concept of convergence in $\bar{w}/\sigma$, which has been deduced here, is the same as the convergence in the second moment ratio, $\overline{w^2}/\bar{w}^2$, recognizing that σ^2 is equivalent to the quantity $\{\overline{w^2} - \bar{w}^2\}$. Thus, the connection between the competitive-equilibrium and shock-free model proposed here and the outcome of dimensional analysis is now made clear.

In relation to this work, nevertheless, convergence in $\bar{w}/\sigma$ should, therefore, yield

$$\mu^*(w_t) \equiv \mu_c \approx \text{constant in time} \qquad (28)$$

since $\mu^*(w_t)$ is, indeed, representative of the ratio $\bar{w}/\sigma$. As a result, Equation 27 reduces to

$$w_t p(w_t) = \mu_c = \text{constant in time} \qquad (29)$$

The above, consequently, implies that as one observes the progress of some wage from w_t to $w_{t+\Delta t}$ (see Fig. 5), the quantity $w_t p(w_t)$ should, in the absence of changes in $\bar{w}/\sigma$, remain constant over time. Moreover, we note that Equation 29 is dimensionless (has no units), and that it involves $\Delta\ln(w_t)$ – i.e. $\Delta w_t/w_t$. From this we conclude that it should not matter whether or not the wages are adjusted for inflation, or whether they are in nominal or in real terms. The non-dimensionality of $\Delta N(w_t)/N(t)$ further suggests that this ratio could represent the fraction of individuals, households, groups, etc.

Now, by virtue of Equation 22, where w_t has been assumed to vary as the average-wage inflation rate, $\zeta(t)$, we assert, using mathematical terminology, that Equation 29 is satisfied along the 'characteristic'

$$\ln(w_t) = \int_t \zeta(t)\,dt = \ln(\bar{w}(t)) + c \qquad (30.1)$$

or

$$w_t = k\bar{w}(t) \qquad \forall k > 0 \qquad (30.2)$$

which are obtained after integrating Equation 22 in time. Here, c is an integration constant and $k \equiv \exp(c)$, implying that $k > 0$ (for real c). With μ_c staying constant over time according to (29), the differential equation

$$\frac{d}{dt}[w_t p(w_t)] = 0 \qquad (31)$$

should, therefore, depict the movement of the wage interval w_t (see Fig. 5), as one observes it along the characteristic described by either Equation 30.1 or 30.2. This should finally yield the transient behaviour of $p(w_t)$, provided that an initial condition, i.e. $p_o(w)$, is supplied, along with the exogenous average-wage growth rate, $\zeta(t)$.

Transformation to a 'stationary frame of reference'

Normally, income distribution data readily available in the literature do not cover firms on an individual basis, nor do they follow each firm's progress over time. These distributions are more generally aggregate compilations of data, presented in cross-sectional form and depicting the wage rates of the labour force at some given time, put together into graphs and/or tables.

To illustrate, we have graphed in Figs. 6 to 8 some income distribution data gathered from tables in the literature (Theil, 1967; *Statistics Canada*, 1994). Figures 6 and 7 are probability density functions, displaying the nominal yearly labour income of 'white' and 'non-white' families, respectively, in the USA for the years 1947, 1955 and 1962, thus covering 16 years (Theil, 1967). Fig. 8, on the other hand, portrays the same for 'all individuals' in Canada for three different years between 1965 and 1994, thus spanning 30 years (*Statistics Canada*, 1994). Note that the earnings distribution data discussed in Section II belong to Canadian 'unattached individuals', which, itself, is a subset and, thus, a fraction of 'all individuals'. This fraction, however, is relatively small in comparison, equalling roughly $\frac{1}{6}$ throughout the years that the data were collected. Furthermore, all data are presented in nominal terms instead of real, owing to the reason given in Section II.

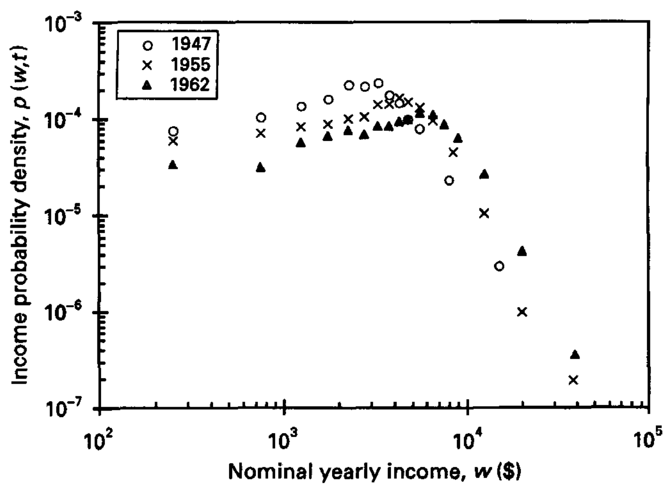

Fig. 6. *Income distribution data for 'white' American households, showing the income probability density,* p(w, t), *plotted against the nominal yearly income,* w, *in current dollars. Data obtained from Theil (1967). Note that the data cover 16 years, from 1947 to 1962*

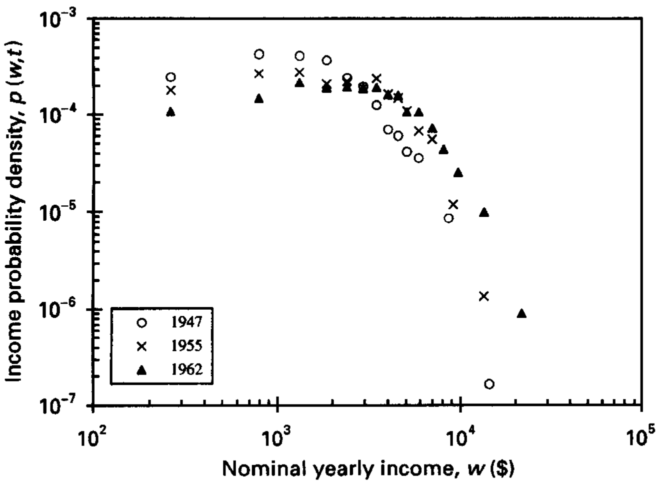

Fig. 7. *Income distribution data for 'non-white' American households, showing the income probability density,* p(w, t) *plotted against the nominal yearly income,* w, *in current dollars. Data obtained from* Theil *(1967). Note that the data cover 16 years, from 1947 to 1962*

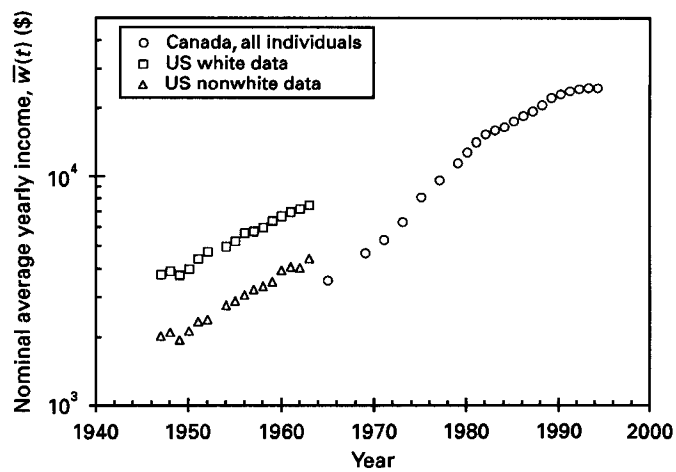

Fig. 9. *Average nominal yearly income,* w̄(t), *in current dollars, plotted against time. The data, which are obtained from Theil (1967) and* Statistics Canada *(1994), correspond with those displayed in Figs 6–8. Note that the US average income increases by about a factor of 2 in 16 years, whereas the Canadian average income increases by about one order of magnitude in 30 years*

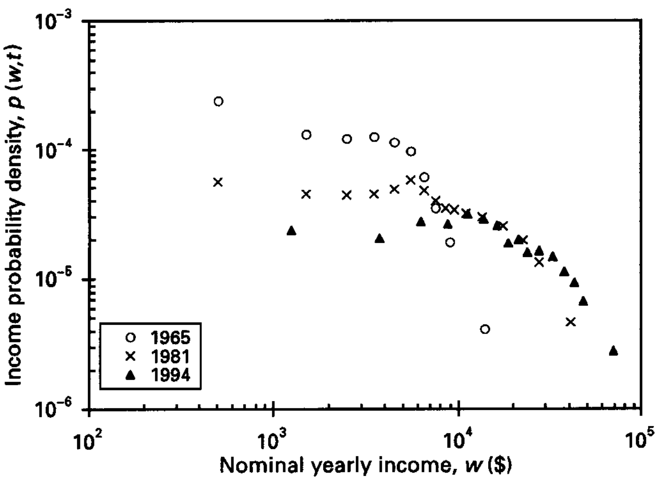

Fig. 8. *Income distribution data for Canada showing the income probability density,* p(w, t), *for all Canadian individuals, plotted against the nominal yearly income,* w, *in current dollars. Data obtained from* Statistics Canada *(1994). Note that the data cover 30 years, from 1965 to 1994*

The tables from which these figures have been extracted provide the fraction of the labour force that falls within a nominal-wage bracket, Δw. Given such tables pertaining to different years, it is not difficult to generate histograms by plotting $\Delta N(w, t)//N(t)$ against the interval mean, w. Here, $\Delta N(w, t)$ is the portion of the labour force that earns a wage of w at time t, and falls within Δw. Note that this is different from the previously described panel representation, $\Delta N(w_t)$, which depicts the wage interval w_t as it is followed in time.

We should mention that, in this analysis, the lower limit of the wage was taken as zero and the upper was computed using the identity $\bar{w}(t) = \Sigma w \Delta N(w, t)/N(t)$.

In addition to these figures, we include Fig. 9 to portray the growth of the overall average wage, $\bar{w}(t)$, over time. If needed, one should be able to compute from this the quantity $\zeta(t)$ using Equation 22.

Given the above, therefore, we redefine the earnings distribution density function, $p(w, t)$, as

$$p(w, t) \equiv \frac{1}{N(t)} \lim_{\Delta w \to 0} \frac{\Delta N(w, t)}{\Delta w} \tag{32}$$

which is identical to Equation 1 and somewhat similar to Equation 19, but different in that now the whole economy is observed from a 'stationary' point of view at an instant in time, instead of a specific earnings interval followed over time.

Realizing that the competitive-equilibrium model developed here (Equation 29 or 31) is based on panel data, which move along the characteristic $\int_t \zeta(t)\,dt$, then direct implementation of typical cross-sectional data to test the model is not permissible. Consequently, a coordinate transformation on Equation 31 is necessary to convert it from a moving frame of reference to a 'stationary' one, after which it could be applied to cross-sectional data. This is accomplished once we substitute for the total-time derivative, which appears there as d/dt, the expression

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \zeta(t)\frac{\partial}{\partial \ln(w)} \tag{33}$$

When applied to (31), therefore, we get

$$\left[\frac{\partial}{\partial t} + \zeta(t)\frac{\partial}{\partial \ln(w)}\right]\{wp(w, t)\} = 0 \quad (34)$$

or

$$\frac{\partial p(w, t)}{\partial t} + \zeta(t)\left[\frac{\partial p(w, t)}{\partial \ln(w)} + p(w, t)\right] = 0 \quad (35)$$

after some rearrangement. The above are now directly applicable to literature data, which are in cross-section form. Also, given that $\zeta(t) = d\ln[\bar{w}(t)]/dt$ from (22), it can be shown by direct substitution (into either (34) or (35)) that a solution of the form

$$wp(w, t) = g(w/\bar{w}(t)) \quad (36)$$

where $g(\xi)$ is some function, satisfies the above. Finally, since Equation 35 is linear, then the general, functional form of the solution offered in Equation 36 is unique.

First of all, Equation 36 represents a special case of (15) in that the product wp depends on the single variable, $w/\bar{w}(t)$, and is independent of the moment ratios. Second, just as discussed in Section II, this type of solution is 'self-similar' and 'time-invariant', which means that wage distributions belonging to different times should all look the same if plotted as $wp(w, t)$ versus $w/\bar{w}(t)$. The outcome of this is that if we were provided with earnings-distribution data, $p(w, t)$, pertaining to different years, such as those shown in Fig. 1 or Figs 6 to 8, then a plot of $wp(w, t)$ versus $w/\bar{w}(t)$ should make all the data collapse on to a single curve described by the function $g(w/\bar{w}(t))$. This, of course, should happen if the economy were to follow the shock-free, competitive-equilibrium model proposed here.

Another useful feature of this class of functions is that from an initial condition, i.e. $p(w, t = 0)$, the function $g(\xi)$ is recoverable. This, in conjunction with the data $\bar{w}(t)\ \forall t \geqslant 0$, should enable one to compute the dynamic behaviour of the wage distribution.

A key question now is what determines the form of $g(\xi)$? For obvious reasons, this function is a general one and, therefore, its shape would depend most likely on the initial market conditions, namely the initial condition $p(w, t = 0)$. We shall elaborate more on this later when we present a model of a simple economy whose initial condition, $p(w, t = 0)$, and hence $g(\xi)$, could easily be deduced. First, however, we plan to test whether or not the competitive-equilibrium model laid out here has any validity.

Evaluating the model

We have, until now, developed a model for a shock-free economy in competitive equilibrium, derived a partial differential equation that describes it (Equation 35), and provided the form of solution that satisfies it (Equation 36). It should, therefore, be possible at this time to utilize data from the literature to verify the model.

Luckily, the procedure for doing this is straightforward. All one needs to do is to plot the quantity $wp(w, t)$ against $w/\bar{w}(t)$ and check whether the data belonging to different years collapse around a single curve. This is in line with the prediction of the dimensional analysis carried out earlier, subject to conditions that certain criteria concerning the moment ratios be met.

On applying this to the data shown in Figs 6 to 8, we obtain Figs 10 to 12. Once again, the solid line present in these figures belongs to a theoretical analysis, which will be explained in Section IV. It should be noted that to obtain

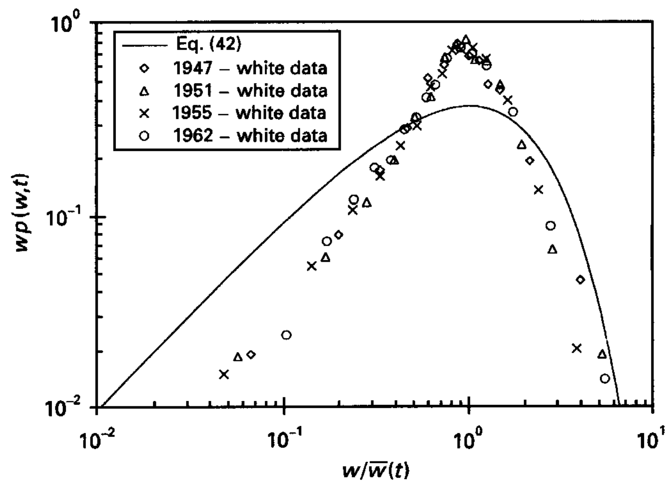

Fig. 10. *Income distribution data of Fig. 6 plotted in transformed coordinates,* wp(w, t), *versus* w/w̄(t)

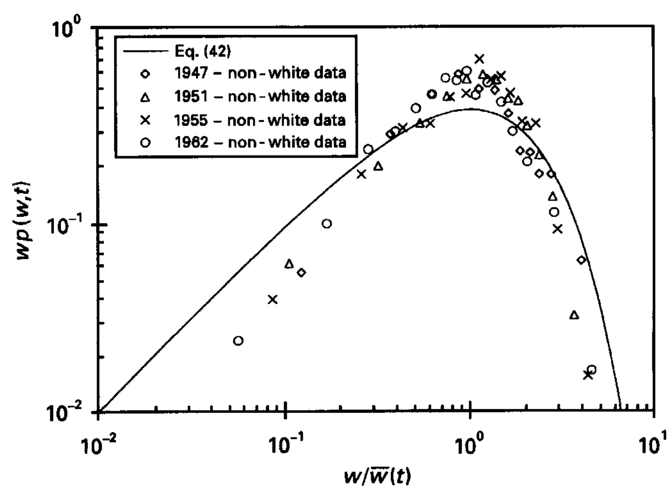

Fig. 11. *Income distribution data of Fig. 7 plotted in transformed coordinates,* wp(w, t), *versus* w/w̄(t)

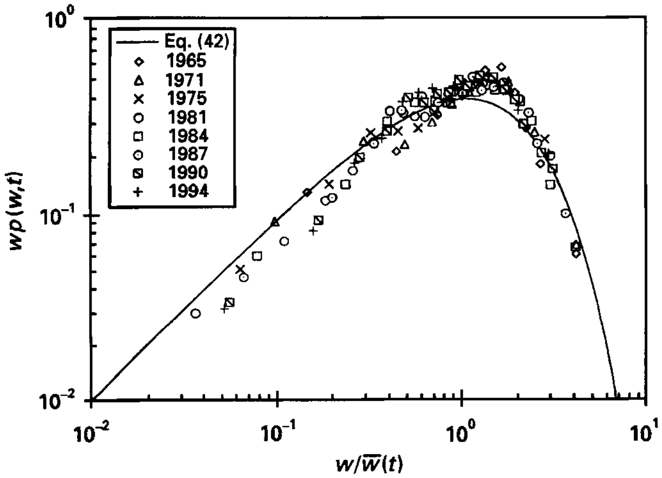

Fig. 12. *Income distribution data of Fig. 8 plotted in transformed coordinates,* wp(w, t), *versus* w/w̄(t)

Figs 10 to 12 (as well as 13 to 16, which will be discussed later), $p(w, t)$ was estimated from the histogram-type data using the previous relation (3), i.e.

$$p(w, t) \approx \frac{\Delta N(w, t)/N(t)}{\Delta w} \tag{3}$$

consistent with (1), or, equivalently, (32).

Interestingly, a pattern compatible with the expected convergence of the data around a single curve emerges in all three cases. Most striking is the tight conformation of the white families' income-distribution data depicted in Fig. 10. Data convergence in Figs 11 and 12, however, is not as compact, but compared with the much scattered points displayed in the corresponding Figs 7 and 8, the outcome is rather promising. In all, considering that no adjustable parameters have been used anywhere in the analysis, the results presented in Figs 10 to 12 are encouraging, and may well attest to the applicability of the competitive-equilibrium model proposed here.

Also interesting is that even though Figs 10 and 12 displays data convergence in the transformed coordinates, apparently around some hypothetical curve $g(\xi)$, the shape of this curve appears to be different in all three cases. The disparity is especially prominent if one were to compare the data in Fig. 10 to those in Figs 3, 11 and 12. This not only suggests that $g(\xi)$ may not be universal, but it could be specific to every different situation. Speculating that the market's macro and micro economies are the root causes for these differences, it would be useful to find a way to determine $g(\xi)$ theoretically, given the economic conditions of the time. For this, we shall propose a model of a simple economy, showing how the function $g(\xi)$ could be derived for that particular case.

Summary and concluding remarks

A model for the dynamic behaviour, but *not* the shape, of the wage distribution in a shock-free economy in competitive equilibrium has been presented. The main outcome of this is a coordinate transformation, which converges the time-dependent wage-distribution data into a single, 'self-similar' and 'time-invariant' curve. This behaviour, under the derived coordinate transformation, is due to convergence in $\bar{w}/\sigma$, which corresponds to convergence of the second moment ratio, $\overline{w^2}/\bar{w}^2$, as deduced from Buckingham's Π theorem.

On applying the transformation to actual Canadian and US income-distribution data, the predicted data convergence around some hypothetical curve, $g(\zeta)$, although not so tight in some of the cases, is indeed observed. This suggests that, in the variety of the situations examined here, the share of earnings among the different wage brackets has not changed considerably over several years, and even decades (see Figs 10 to 12). In other words, the income distributions considered here, as well as in Section II, have already converged to some sort of a steady state. Surprisingly, this contradicts what much of the relevant literature emphasizes – namely that, in terms of earnings, there has always been a constant movement away from equality. Nevertheless, it must be recognized that our results are based on intersectoral earnings data, which were collected from within relatively narrow segments of the population, where time-dependent variations in equality might have been paled by other events. In a broader sense, however, where the earnings of different sectors of the population are compared against one another via suitable deflators, the situation might change, and variations in equality in terms of real earnings, if they exist, could prove to be more prominent.

It is difficult, none the less, to speculate at this time on how our findings apply to earnings across sectors or countries, because this was not the issue of concern here. Moreover, we cannot elaborate on whether or not our model is missing any important element, except that the evidence reported here clearly points towards the existence of a steady state in the distribution of the quantity $wp(w, t)$ (see graphs), and the data used here were obtained from well-known, published figures and statistics, all of which are publicly accessible through data banks and reference materials.

IV. THE EARNINGS DISTRIBUTION IN AN ECONOMY WITH PERFECT LABOUR MOBILITY

A model for $g(\xi)$

The competitive-equilibrium model developed so far to describe the dynamic behaviour of the earnings distribution requires an initial condition, such as the earnings distribution density function at time $t = 0$ (i.e. $p(w, t = 0)$). Using this, together with the coordinate transformation outlined earlier, one should be able to retrieve the function $g(\xi)$, which, in turn, is sufficient to yield the time-wise evolution of the earnings distribution function. All this is, of course, feasible if the distribution function were self-similar under the suggested coordinate transformation.

The objective here is to derive a simple model that leads to an initial condition for the earnings distribution. Evidently, different economic conditions are expected to give rise to different $g(\xi)$s. For our purpose, we choose to investigate a simple example: (i) the industry is 'large', employing many workers; (ii) the industry is homogeneous in that it allows its workers to be perfectly mobile across the wage brackets or intervals; and (iii) the industry covers many wage intervals by offering a wide range of wages. A good example of such an industry is restaurants. When taken jointly, restaurants employ mostly unskilled labour (as waiters, cashiers, busboys, etc.), which are quite mobile, and makes available to them a wide range of wages spanning from the minimum wage in fast-food chains to the much higher in more elaborate places.

Now, suppose that the industry comes into being in time $t = 0$, when it hires a total labour force of $N(t = 0)$ workers. Based on the assumption that the labour force is perfectly mobile across the wage brackets, the workers could then be placed *randomly* into the different, available ones. This, in a sense, could also signify equality of the workers in the context of Theil's work (1967) because every wage bracket is equally accessible to each of the workers.

If we let $\Delta N(1, t = 0)$, $\Delta N(2, t = 0)$, etc. be, respectively, the number of workers place at time $t = 0$ into wage brackets offering average amounts of \$1 per unit time, \$2 per unit time and so on, it follows that

$$N(0) = \sum_{w=0}^{\infty} \Delta N(w, t = 0) \tag{37}$$

which is Equation 18 evaluated at time $t = 0$. The upper limit of the sum, which certainly is unrealistic, is put there purely for mathematical convenience, bearing in mind that $\Delta N(w, t = 0)/N(0)$ approaches zero fast as w increases.

The assumption that each wage bracket is equally accessible to each worker raises the question: why shouldn't all the labour force be concentrated into the highest wage bracket, if the industry is providing it? The reason is the resource constraint, which is given by

$$\bar{w}(0) = \sum_{w=0}^{\infty} \frac{w\Delta N(w, t = 0)}{N(t = 0)} \tag{38}$$

where it is assumed that this large industry, at its inception, offers a limited *average* wage of $\bar{w}(0)$. Moreover, why shouldn't all the workers compete for the average wage of $\bar{w}(0)$ instead of accepting a distribution around $\bar{w}(0)$? This is because workers will vie for higher-paying jobs, and while some will get them, the constraints imposed by (37) and (38) will force others to accept the lower-paying ones. As a result, a distribution of wages forms around $\bar{w}(0)$.

To approach this problem quantitatively, we define $\Omega(\Delta N(1, t = 0), \Delta N(2, t = 0), \ldots)$ to be the total number of ways that the full labour force of $N(0)$ workers could be distributed randomly into groups comprising $\Delta N(1, t = 0)$, $N(2, t = 0)$, etc. In relation to our assumptions, this is

$$\Omega(\Delta N(1, t = 0), \Delta N(2, t = 0), \ldots) = \frac{N(0)!}{\Delta N(1, t = 0)!\, \Delta N(2, t = 0)! \ldots} \tag{39}$$

where random placement in this context is synonymous with perfect worker mobility.

The next step is to use the method of Lagrange mulitpliers to maximize $\ln(\Omega)$ with respect to $\Delta N(w, t = 0)$, while subjecting it to the constraints of (37) and (38). The result would then be the *most probable* wage distribution in this large industry that allows its workers to be perfectly mobile across the wage brackets

We have, in the interest of space, omitted the calculations because they are quite common in the literature, particularly those that utilize maximum-entropy concepts (Theil, 1967; Golan, 1994 and references therein). In all, it is not difficult to show that the most probable wage distribution, under the given assumptions and constraints, follows

$$p(w, 0) = \frac{1}{\bar{w}(0)} \exp\left(-\frac{w}{\bar{w}(0)}\right) \tag{40}$$

upon using the relationship between $p(w, t = 0)$ and $\Delta N(w, t = 0)$ given in Equation 32. Note that this corresponds to a maximum in Theil's index of equality (Theil, 1967), thereby confirming the implications of perfect mobility in this model.

Next, we rewrite (40) as

$$wp(w, 0) = \frac{w}{\bar{w}(0)} \exp\left(-\frac{w}{\bar{w}(0)}\right) \tag{41}$$

which resembles Equation 36. From this it follows that the above could very well represent the initial condition for a special type of self-similar function, $g(w/\bar{w}(t))$. Therefore, if this large, wide-wage range and homogeneous industry under consideration here were to flourish in the aforementioned competitive-equilibrium and shock-free economy, the income distribution density function should then evolve according to

$$wp(w, t) = \frac{w}{\bar{w}(t)} \exp\left(-\frac{w}{\bar{w}(t)}\right) \tag{42}$$

This, it should be noted, is specific to the case modelled here, and varying market conditions should expectedly give rise to different forms for $g(w/\bar{w}(t))$.

Comparison of model and data

We have derived a model for the distribution of earnings belonging to an economy in which the labour force is perfectly mobile across the wage brackets. The result, given by Equation 42, is observed to satisfy the rules of self-similarity and time-invariance, which were explained earlier.

Returning now to Figs 3, 10 to 12, as well as 13 to 16 we have, for contrast, included the behaviour of Equation 42 as well. This is portrayed by the solid line in those figures.

First, it is clear in all cases that the data do not follow exactly this model for $g(\xi)$. The differences, notwithstanding, are not so large either, especially in Figs 11 and 12. Qualitatively, to say the least, the agreement is worthy of attention, considering that no adjustable parameters have been implemented anywhere.

Second, we include Table 1 for a quantitative comparison between data and the perfect-mobility model Equation 42, both displayed in the relevant figures. This table displays for each of the figures the standard error of residual, SER,

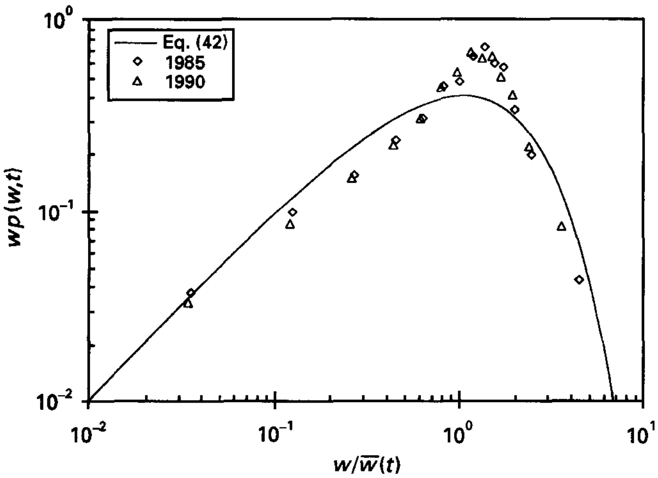

Fig. 13. *Canadian men's income distribution data plotted in transformed coordinates*, wp(w, t), *versus* w/w̄(t). *The 1985 figures are in terms of 1990 dollars* (Statistics Canada, *1990*)

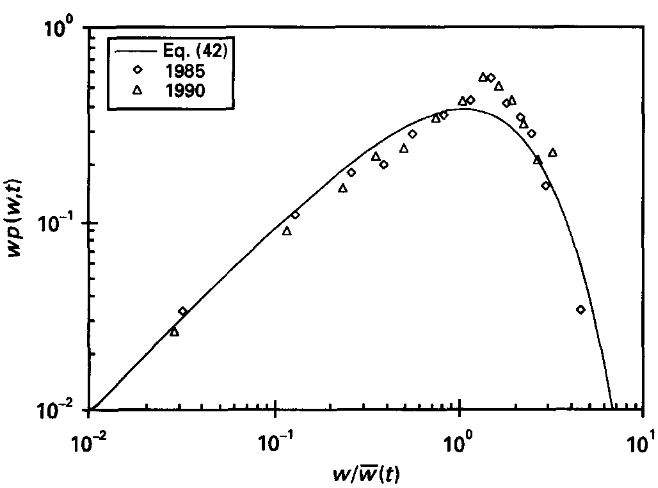

Fig. 14. *Canadian women's wage distribution data plotted in transformed coordinates*, wp(w, t), *versus* w/w̄(t). *The 1985 figures are in terms of 1990 dollars* (Statistics Canada, *1990*).

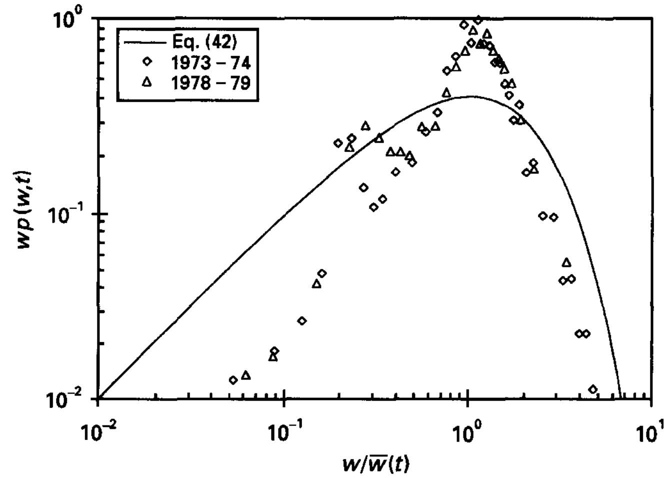

Fig. 15. *Australian men's income distribution data plotted in transformed coordinates*, wp(w, t), *versus* w/w̄(t). *Data for 1973–74 obtained from* Australia Yearbook (*1975–76*) *and 1978–79 data from* Australia Yearbook (*1981*). *Note that the mean incomes,* w̄(*1973–74*) *and* w̄(*1978–79*), *were 5710 and 10 170 Australian dollars, respectively*

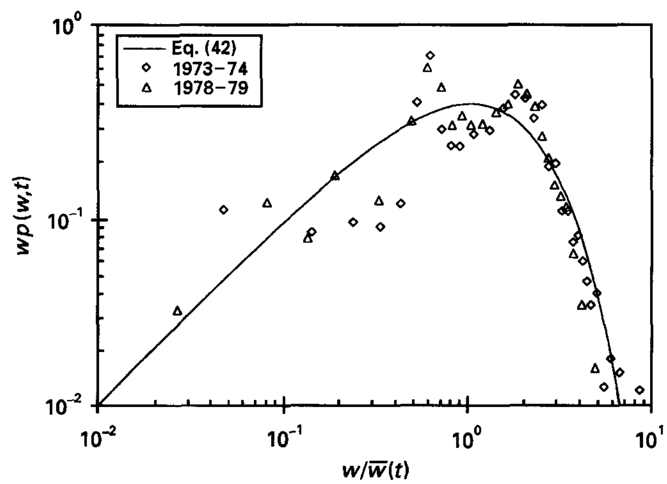

Fig. 16. *Australian women's income distribution data plotted in transformed coordinates*, wp(w, t), *versus* w/w̄(t). *Data for 1973–74 obtained from* Australia Yearbook (*1975–76*) *and 1978–79 data from* Australia Yearbook (*1981*). *Note that the mean incomes,* w̄(*1973–74*), *and* w̄(*1978–79*) *were 2160 and 4720 Australian dollars, respectively*

Table 1. *Standard error of residual, SER, between data and Equation 42. See Equation 43 for details*

Figure	Description	n	SER
3	Unattached Canadian individuals	112	0.538
10	White US households	61	0.629
11	Non-white US households	58	0.524
12	All Canadian individuals	114	0.211
13	Canadian men	28	0.339
14	Canadian women	26	0.247
15	Australian men	64	0.653
16	Australian women	58	0.241

which is defined here by

$$\mathrm{SER} \equiv \sqrt{\frac{1}{n}\sum_i \{\ln(g_i) - [\ln(\xi_i) - \xi_i]\}^2} \qquad (43)$$

where n is the number of observations, and g_i and ξ_i, respectively, are $wp(w, t)$ and its corresponding $w/\bar{w}(t)$. The term $\{\ln(g_i) - [\ln(\xi_i) - \xi_i]\}$, therefore, characterizes the residual between empirical data, $\ln(g_i)$, and the natural log of Equation 42.

Based on visual observation, as well as on the information displayed in Table 1, we conclude that, on comparing the US 'non-white' households' labour earnings (Fig. 11) with the 'white' families' (Fig. 10), the non-whites' data fall closer to the homogeneous-worker model. A possible explanation for this is that the US non-white workers may have, at that time, had a tendency to be employed by a more homogeneous, perhaps low-skilled type industry, which enabled a higher cross-interval mobility. The white labour force, on the other hand, might have been composed of a more heterogeneous group, consisting of both skilled and unskilled labour, and thus maintaining a lower cross-wage mobility.

Moreover, a comparison of the data points belonging to 'unattached individuals' (Fig. 3) with those of 'all individuals' in Canada (Fig. 12) reveals that the former lie farther away from the theoretical model. This, perhaps, is due to some additional constraints related to labour force mobility, which might be associated with being 'unattached'. As mentioned earlier, the unattached set comprises a relatively small subset, equal in fraction to roughly 15% of the total.

To reinforce the hypothesis of labour mobility, we display in Figs 13 and 14 the earnings distributions, in transformed coordinates, of Canadian 'men' and 'women' (*Statistics Canada*, 1990), and compare them with Equation 42. From Table 1 and a visual examination of the figures, a closer association between Equation 42 and the women's earnings distribution data is established. A possible explanation for this is that the women's labour force is more homogeneous (perhaps mostly on the low-skilled side) and, thus, more mobile across the wage intervals.

A similar conclusion is again reached if we were to compare the men's and women's earnings in Australia. Once more, judging from Figs 15 and 16, which display these data for the years 1973–74 and 1978–79, the self-similarity rule is respected to a reasonable degree for both sexes (there is, however, some scatter in the tail end of the data distribution in Fig. 16). In addition, as indicated by the error analysis in Table 1, the women's data lie closer to Equation 42, which implies a more mobile labour force, perhaps due to it being or treated as mainly low-skilled.

It follows from the above arguments and numbers, therefore, that the economy treats women's labour, on aggregate, as more homogeneous than the men's. This could arguably attest to discrimination against women in the countries examined, because it signifies an economy that fails to recognize the heterogeneity in skills and levels of effort, which is inherent to any labour force.

Summary and concluding remarks

On extending the competitive-equilibrium and shock-free economic model of Section III to account for a labour force that is perfectly mobile across wage intervals, we were able to derive a particular shape for the earnings distribution specific to this type of economy. The behaviour of this distribution is, again, self-similar and time-invariant, like the ones described in Sections II and III. The shape of it, however, which follows Equation 42, reflects a labour force that is perfectly mobile across different income intervals. Interestingly, upon comparing the results of this simple theoretical model with some of the earnings distribution data discussed throughout this work, we observe fair agreement, at least qualitatively, when both, data and model, are presented in the $wp(w, t)$ versus $w/\bar{w}(t)$ coordinate system (compare the data points and the solid line in Figs 3, 10 to 12, and 13 to 16).

Altogether, the model presented here is simple and straightforward, and, in some of the cases, it seems to agree reasonably well with data, making no use whatsoever of adjustable parameters for fitting purposes. Notwithstanding, this work remains far from complete since not only does it concentrate on a very special type of economy, but it also does not incorporate any elaborate econometric procedures – ones that go beyond the elementary error analysis outlined in Section IV – to verify the model. Conducting such econometric tests at this point is beyond the scope of this work and, therefore, it is best left as part of future work.

V. CONCLUSIONS

An investigation of the time-wise dynamic evolution of the earnings distribution within sectors of a population was the focus of this work. The analysis was carried out in three parts. First, the method of dimensional analysis, which is more common and popular in engineering practices, was used to show that earnings distributions are self-similar and time-invariant under certain conditions and a special coordinate transformation. The conditions leading to these entail the stationarity or time convergence of the moment ratios. On applying the method to a sector of the Canadian population, these stationarity criteria were observed to be satisfied, and, as predicted, the expected time-invariant and self-similar features did indeed emerge.

A model of a shock-free economy in competitive equilibrium was then constructed to explain the outcome of the dimensional analysis, and to provide some insight into its results. This not only confirmed the coordinate transformation and stationarity conditions generated earlier, but it also unveiled some of the hidden characteristics, in terms of data convergence in the transformed coordinate system that were embedded within the analysed data. In this regard, it is important to recognize that the model, unlike most major relevant ones, does not incorporate any adjustable parameters for fitting purposes. Moreover, at least within the limits of the intersectoral earnings data considered here, our findings do not lend any firm support to the notion that inequality has been rising over the past few years.

The model was finally extended to account for perfect labour mobility across wage intervals, and to assess its consequences on the earnings distribution. For analysis, the maximum entropy approach, which is a relatively common analytical tool in engineering and physics, was utilized. The outcome of this is summarized by Equation 42, which is also portrayed by the solid line in Figs 3, 10 to 12 and 13 to 16. The nearness of this line to a given set of earnings distribution data, therefore, reflects the degree of cross-interval mobility within the labour force represented by those data. Among the conclusions reached here is that, in general, women and non-white labour forces seem to be more mobile across wage intervals. This could imply discrimination, as explained at the end of Section IV.

In so far as policies are concerned, the implications of this work, owing to its nature, lean more towards policy implementation than design. In this regard, Equation 27, which forms the basis of the model in Section III, suggests some ways for implementing policies in a competitive economic setting. First, the absence of 'resistance' or 'inertia' here indicates that once a policy is implemented (i.e., through changes instituted in the term $\mu^*(w_t)$, such as by increasing productivity and/or varying commodity prices) its effects must be realized immediately. If this were not the case, then the policy might be ineffective, and should be either revised or discarded.

Second, the absence of 'diffusion' terms (higher-order derivatives in w) in Equation 27 or 34 implies that the effects of policies remain localized. For instance, a policy that focuses on a specific income bracket (again, this could be accomplished by varying the prices, productivity and/or wages that belong to that income bracket, thus directly affecting $\mu^*(w_t)$) should influence only that particular bracket. In other words, a policy that targets the poor should affect only the poor and vice versa, even in the long run. This coincides with certain issues related to poverty alleviation through decentralization of the governance structure, where the general consensus is that policies targeting poverty alleviation are more effective when implemented locally rather than centrally Bhardan, 1996). It appears, therefore, that there are no trickling effects in a competitive economy, unless redistributive measures, such as by taxation or other means, are taken.

As for recommended future research in this direction, removing or relaxing some of the simplifying assumptions that have led to the models described in Sections III and IV would constitute a good start. In Section III, for example, one could generalize the derivation of Equation 27, and consequently (34), by taking a wide, instead of a homogeneous (see Section III), array of prices and products to represent the local economy within an income bracket. Although we now believe that this should not change the final outcome, a formal proof could help eliminate any existing doubt. Moreover, extending the model to a non-equilibrium setting, where, for instance, the quantity supplied does not equal the quantity demanded, which is contrary to Equation 25, could invoke some interesting issues.

With regard to the model described in Section IV, the assumption of perfect mobility across wage intervals, on which Equation 42 is founded, could be generalized by imposing certain mobility-related constraints. A recommended way for doing this analytically is to follow the line of reasoning that led to Equation 42, but somehow incorporate barriers to mobility in the analysis. Admittedly, this is easier said than done, but the use of the method of maximum entropy principle could help render this problem tractable. A model that emerges from such a study could potentially shed light, given only the time-dependent income distribution data, on where in the aggregate economy the labour force is mobile and where it is not. Consequently, this could help generate policies that work to either promote labour-force mobility or resist it.

Altogether, aside from the policy implications described above, this work contributes to the economics literature in at least three ways. First, it reintroduces the important analytical tool of dimensional analysis, which is basic to engineering but not to economics, to investigate the dynamic behaviour of earnings distributions. Second, it offers a quick and simple way for determining how close an economy comes to being, within a given time frame, in competitive equilibrium and relatively free of impacts of shocks. And third, it provides a straightforward approach for quantitatively assessing the degree of mobility of a labour force within a sector of an economy.

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