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The Relative Valuation of an Equity Price Index[1]

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A new approach for the relative valuation of an equity price index is presented. The method is based on a coordinate transformation or mapping, which enables one to weigh the index against the aggregated earnings and GDP. This, therefore, gives rise to the notion of relative valuation between the index, the earnings and the GDP. A practical demonstration of this is then provided for the US, UK and Japan economies and some of their major equity indices, namely the S&P500, FTSE100 and TOPIX, respectively.

Another potential application of the above is also discussed, which relates to forecasting the GDP. This stems from the assumption that the expected GDP, one year ahead from today, is readily priced in today's interest rates. The method is further applied to computing duration. This is shown to circumvent the difficulties that are generally associated with calculating the parameter.

1 Introduction

Relative valuation is a generic term that refers to the notion of comparing the price of an asset to the market value of similar assets. In the field of securities investment, the idea has led to important practical tools, which could presumably spot pricing anomalies. Over time, these tools have become instrumental in enabling analysts and investors to make vital decisions on asset allocation.

In equities, the concept separates into two areas—one pertaining to individual equities and the other to indices. The most common methodology for the former is based on comparing certain financial ratios or multiples, such as the price to book value, price to earnings, EBITDA to enterprise value, etc., of the equity in question to those of its peers (see, for instance, Barth *et al*. 1998, D'Mello *et al*. 1991 and Peters 1991). This type of approach, which is largely popular as a strategic tool in the financial industry, is mainly statistical and based on historical data.

For an equity index, however, the above fails mainly because it is difficult to group indices into peer groups. Consequently, relative valuation here is generally carried out by comparing the index's performance to economic and market fundamentals, which may include GDP growth, interest rate and inflation forecasts, as well as earnings growth, among others. This style of comparison is popular among practising economists in their attempt to rationalise the connections between the equity markets and the economy.

The above approach also has its faults, however—one being that, even if the fundamentals were known, there appears to be no consensus methodology, as the procedures that are generally implemented tend to be subjective, ad hoc and dependent on personal style. Thus, it would be useful to devise a new approach to enable one to add some objectivity to the process.

In constructing such a framework here, the classical equity valuation models are first summarised, after which the role of the equity risk premium and how it fits in are clarified. A couple or so simple propositions are then brought in to help facilitate the process. The use of this new method is later demonstrated by (1) suggesting other potential applications, such as forecasting the GDP and calculating duration and (2) incorporating it as a relative-valuation tool. It should be noted that, owing to the nature of the approach, there is no need for any detailed statistical testing, as conclusions can be drawn simply by visual examination of graphs and charts alone.

2 A background on equity valuation

Since the classical models of equity valuation are covered well in the literature, it would be repetitive to discuss them here in any depth. Nevertheless, it is still necessary to go over some of the assumptions and limitations that underlie these models, as they comprise part of the foundation upon which the new model for relative valuation is based.

2.1 The classical models of equity valuation

In the classical theory of equity valuation, three relationships dominate. They are:

$$\frac{S_f(t) - S(t)}{S(t)} + \frac{\delta_f(t)}{S(t)} = R_M(t) \tag{2.1}$$

$$\frac{\delta_f(t) - \delta(t)}{\delta(t)} + \frac{\delta_f(t)}{S(t)} = R_I(t) \tag{2.2}$$

and

$$\frac{E_f(t)}{S(t)} = R_F(t) \tag{2.3}$$

where $S(t)$ and $\delta(t)$, respectively, are the price and dividends at time t, while $S_f(t)$, $\delta_f(t)$ and $E_f(t)$ signify the 'expected' price, dividends and earnings (after interest and tax, but before dividends). These are yearly expectations, generated for one year ahead from today.

With regards to the above, note that, while Equation 2.1 is an identity, with $R_M(t)$ denoting the expected total rate of return, Equations 2.2 and 2.3 represent valuation models, namely, Gordon's Growth Model[2] and the discounted-cash-flow (DCF) relationship,[3,4] respectively, with $R_I(t)$ and $R_F(t)$ being their expected discount rates. The equity risk premium is discussed briefly in the next section, after which the derivation of the relative valuation model will be carried out.

2.2 The equity risk premium

Owing to its importance in the area of equity investment, the equity risk premium has always attracted attention from academics and practitioners. Countless papers have been written so far on the subject, each proposing a reason for why the risk premium should exist, what it depends on and/or how large it should be. Although many of these works present conflicting theories and/or conclusions, all concur unanimously that the risk premium is a result of uncertainties. It is not the concern here to discuss what causes these uncertainties. These uncertainties simply exist, have always been and will remain to be around as long as no one can predict accurately what the future—near-term or far—holds for the economy and markets.

What is relevant here is how does the equity risk premium, as a parameter, get integrated into valuation? By definition, the risk premium is the difference between the rate of return or discount rate, which could be any of the ones appearing in Equations 2.1–2.3 above, and some 'risk-free' rate.[5] As to what discount rate and risk-free rate one should use is another matter, which, again, shall be left out here. Rather, what is important is that under total and unconditional absence of all uncertainty—past, present and future—the risk premium would not exist, so that all the rates of return that appear in Equations 2.1–2.3 become equal to the 'true' risk-free rate, which itself would remain constant and free of volatility.[6] This, therefore, leads to Proposition 1, which may be expressed as:

Proposition 1 In the absence of all uncertainty and change—past, present and going forward—all risk premiums become zero.

Thus, what entails the above proposition is that all arbitrage opportunities between different types of securities disappear. For instance, equity and fixed income instruments will yield the same, as the yield curve flattens and becomes horizontal. In this instance, therefore, all yields will equal b^*, where b^* symbolises the 'true' risk-free rate. Moreover, in the absence of the risk premium, all rates of return (or discount rates) in Equations 2.1–2.3 will also equal b^*.

In addition to the above, the golden rule of economics enters also, so that

$$\frac{d \ln G}{dt} = \left(\frac{\partial \ln G}{\partial t}\right)_{b=b^*=constant} = b^* \tag{2.4}$$

where G is the level of the nominal GDP and b is the interest rate, which is set constant at b^*. Finally, all forecasts in 2.1–2.3 above—i.e. $S_f(t)$, $\delta_f(t)$ and $E_f(t)$—become identical to their real-time counterparts, $S(t+1)$, $\delta(t+1)$ and $E(t+1)$, respectively, realised a year later at $t+1$.

With Proposition 1 in place, Proposition 2 may now be stated as:

Proposition 2 Under Proposition 1, the golden rule applies also to the rate of growth in equity earnings.

Proposition 2 basically unites the golden rule, as it relates to the GDP in Equation 2.4, to equity earnings as well. This is possible under the above circumstances because equity earnings, or profits, comprise a subset of the GDP and, in the absence of arbitrage, all subdivisions within the GDP must yield at the same rate.

Quantitatively, this is expressible by

$$\left(\frac{\partial \ln E}{\partial t}\right)_{b=b^*=constant} = b^* \tag{2.5}$$

where E is the equity earnings. Thus, under Propositions 1 and 2, with all rates of return in 2.1–2.3 being equal to b^*, as well as the forecasts of S, δ and E remaining identical to their real-time counterparts a year later, Equation 2.5 may be applied to 2.3 to give:

$$\left(\frac{\partial \ln E}{\partial t}\right)_{b=b^*} = \left(\frac{\partial \ln S}{\partial t}\right)_{b=b^*} = b^* \tag{2.6}$$

since, in this case, the discount rate, R_F, also equals b^*.

The implication of Equation 2.6, which states that, subject to the conditions imposed above, the golden rule applies as well to the equity price, $S(t)$, is significant. This is because, upon first using the approximation[7]

$$\left(\frac{\partial \ln S}{\partial t}\right)_{b=b^*} \approx \frac{S(t+1) - S(t)}{S(t)} \tag{2.7}$$

then substituting 2.6 and 2.7 into 2.1 and, finally, setting the rate of return, $R_M(t)$, equal to b^*, all in the absence of the risk premium, the dividend yield, $\delta(t+1)/S(t)$, tends to zero. This simply suggests that, in a world with no uncertainty and change, and, hence, no risk premiums, the investor will not demand any dividend yield.[8]

Therefore, do markets pay and/or investors demand a positive dividend yield because of uncertainties? This, inevitably, points to the much debated issue of the dividend puzzle, along with its link to the equity risk premium, both of which will be left out here as they are not relevant to this work, but, nonetheless, whose details may be found elsewhere (Cohen 2002). Notwithstanding, the above conclusions do lead to the next step, which is to develop a model for the relative valuation of an equity price index.

3 A model for the relative valuation of an equity price index

The new model for relative valuation is constructed here in two ways—one focusing on equity (Section 3.1) and the other on the fundamentals, namely GDP and equity earnings (Section 3.2). The latter two occupy the same section because their underlying principles happen to be the same. The final results will then be united to present the relative valuation measures.

3.1 The equity model

Beginning here with Equation 2.6, which states

$$\left(\frac{\partial \ln S}{\partial t}\right)_{b=b^*} = b^* \tag{2.6}$$

it follows that $\ln S$ could be written as a function of time, t, as well as b^*—i.e.:

$$\ln S = \ln S(b^*, t) \tag{3.1}$$

In the above, holding the discount rate constant at b^* clearly imposes a severe constraint on S. This, however, may be relaxed by proceeding as follows. Very briefly, in place of writing

$\ln S(b^*, t)$ as done in 3.1, it shall be expressed as

$$\ln S = \ln S(b, t) \tag{3.2}$$

which generalises S to account for a time-variable discount rate, $b = b(t)$, instead.

The rationale behind Equation 3.2 is that the effects of the market, and the economy in general, on S are presumed to enter separately through two fundamental elements, one which is b and the other which comprises everything else that falls outside the reign of b. As the second variable appears as time, t, it renders Equation 3.2 general and, hence, together with $b(t)$, it should capture all the economic and market effects on the price, S. In other words, expressing S in the form of 3.2 effectively removes all the restrictions imposed on it earlier in Equation 3.1.

In view of the above, the *total* time differential of Equation 3.2, subsequently, becomes:

$$\frac{\Delta \ln S(b, t)}{\Delta t} = \left(\frac{\partial \ln S}{\partial t}\right)_b + \left(\frac{\partial \ln S}{\partial b}\right)_t \frac{\Delta b}{\Delta t} \tag{3.3}$$

where Δ denotes time-wise differential—i.e. $\Delta b \equiv b(t + 1) - b(t)$. While the first partial differential—i.e. $(\partial \ln S/\partial t)_b$—has been shown to be equal to b (see Equation 2.6), the second, $(\partial \ln S/\partial b)_t$, is simply the stock duration, which is the sensitivity of the price to changes in b at some given point in time.

Being an 'exact differential', therefore, the two components in Equation 3.3 are coupled to each other via:

$$\left(\frac{\partial}{\partial b}\left(\frac{\partial \ln S}{\partial t}\right)_b\right)_t = \left(\frac{\partial}{\partial t}\left(\frac{\partial \ln S}{\partial b}\right)_t\right)_b \tag{3.4}$$

Since, by virtue of 2.6, the left-hand side of the above is 1, the above equation simplifies to:

$$\left(\frac{\partial}{\partial t}\left(\frac{\partial \ln S}{\partial b}\right)_t\right)_b = 1 \tag{3.5}$$

which may be integrated twice to yield a general solution of the form:

$$\ln S = bt + \alpha_0 + \alpha_1 b + \tilde{\Psi}(b)$$

where α_0 and α_1 are integration constants and $\tilde{\Psi}(b)$ a yet unknown function of *only* b.

Alternatively, the above may be recast into:

$$\ln S - bt = \Psi(b) \tag{3.6}$$

where $\Psi(b)$ is another function of b. The latter representation conveniently absorbs $\tilde{\Psi}(b)$, α_0 and $\alpha_1 b$ into a single function, $\Psi(b)$.

It thus follows from 3.6 that plotting the quantity $\ln S - bt$ against b should, in theory, produce a *single* curve, depending *only* on b. This transformation, as a result, brings in all the effects of time on $\ln S - bt$ through b. A schematic illustration of this is presented in Figure 1, where a mapping of S versus b into $\ln S - bt$ versus b is shown to introduce some type of regularity to a relatively disordered graph.[9,10]

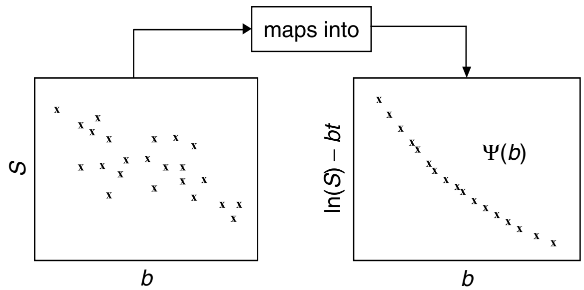


Figure 1: Schematic of the convergence of data points under the proposed coordinate transformation

In light of the derivation so far, it is necessary to mention two points. First, even though Equation 3.6 is extracted from what appears to be too theoretical an approach, it is indeed easy to apply to real situations and, also, as it shall be demonstrated shortly, it does possess other practical uses too. Second, questions relating to what b is—i.e. what interest rate should one use here—have undoubtedly been raised by now. The answer to these, as it will turn out later, happens to be straightforward. Beforehand, however, the same logic is applied next to both the nominal GDP and earnings, as similar transformations are derived.

3.2 Applications to GDP and earnings

It is well accepted that movements in the equity price index are tied closely to corporate earnings and, even more generally, to the economy. Common sense further dictates that a bull market comes typically with a strong economy and a bear market with a weak one. An explanation for this correlation is that the market comprises a subset of the economy—i.e. corporate earnings constitute a (small) fraction of the GDP. This, therefore, should enable one to derive a GDP relationship analogous to the one for equity, as well as for corporate earnings.

Before going into that, however, we need to introduce, with the help of the DCF,[11] a couple of analogies to the equity price index. For this, define V_G and V_E as the 'values' associated with the nominal GDP and corporate earnings, respectively.[12] Therefore, under Propositions 1 and 2, V_G could be represented by

$$V_G \equiv \frac{G_f}{b^*} \tag{3.7a}$$

and V_E by

$$V_E \equiv \frac{E_f}{b^*} \tag{3.7b}$$

where G_f and E_f, respectively, are the time-t expectations of the nominal GDP and corporate earnings one year ahead, at $t + 1$. Hence, with b^* analogous to the discount rate in a 'constant' world, the DCF valuation model is being imposed on the economy as well. It should further be stressed that the one-year-ahead nominal GDP, i.e. $G(t + 1)$, will from now on be implemented instead of the expected for no reason other than convenience, as it shall be assumed that the two converge in an information-efficient economy. For the expected corporate earnings, E_f, on the other hand, Datastream's aggregate I/B/E/S forecasts will be presumed sufficient for the purposes of this work.

Now, with the above analogy in place, it is simple to demonstrate that upon relaxing the constraint on b^* (i.e. replace b^* with b, as it was done in going from Equation 3.1 to 3.2), the same rules that govern the price index should apply as well to V_G and V_E, yielding expressions similar to Equation 3.6, but with V_G and V_E substituted for S. This, consequently, leads to:

$$\ln V_G - bt = \Phi(b) \tag{3.8a}$$

and

$$\ln V_E - bt = \Xi(b) \tag{3.8b}$$

where, as before, $\Phi(b)$ and $\Xi(b)$ are functions of only b.

It should be emphasised that, even though the same transformation that presides over the equity model applies to here as well, the functions $\Phi(b)$ and $\Xi(b)$ may not necessarily be the same as $\Psi(b)$. A comparison of these will be made later; however, certain issues that this raises, namely of the interest rate, 'reversibility' and 'structural or regime shifts', must be addressed beforehand.

3.2.1 Reversibility and structural shifts The representations for the equity price index, GDP and earnings, which are provided in Equations 3.6 to 3.8, lead to the important notions of 'reversibility' and 'structural shifts'. Recognising that structural shifts tend to alter the behaviour of the economy and the markets, an important objective here, as in any economic and financial analysis, would thereby consist of defining ways for detecting and, possibly, classifying them.

To carry this out, observe that $\ln S - bt$, $\ln V_G - bt$ and $\ln V_E - bt$ must depend solely on b via the functions $\Psi(b)$, $\Phi(b)$, and $\Xi(b)$, respectively. The effect of time, as mentioned earlier, enters indirectly through b. Whether or not this functional dependence of Ψ, Φ and Ξ on b is the same in all situations is not of concern now, but, eventually, it shall be dealt with.

An important by-product of such dependence is the concept of 'reversibility', which may be explained via Figure 1 as follows. In reference to this figure, it is noted that, while the unmapped price, S, varies with both b and t and leads to a scattered plot of S versus b, the mapped counterpart changes only with b. This implies that if, for example, the price is S_1 at time t_1, when b equals, let us say, 5%, then at a later time t_2, when b reverts back to 5%, the transformed parameters, $\ln S_1 - bt_1$ and $\ln S_2 - bt_2$, calculated at both times, t_1 and t_2, respectively, must reach the same value again, regardless of the path taken from 1 to 2. This, of course, should apply to V_G and V_E as well, simply by virtue of Equations 3.8a and 3.8b.

Alternatively, a structural or regime shift implies the contrary. If, for instance, a transformed plot produces notably disparate lines, then it is likely that a structural shift has occurred somewhere in between. Schematically, a structural shift is exemplified in Figure 2, where mapping S versus b into $\ln S - bt$ versus b over a given time frame leads to distinctive characteristic patterns. In a similar manner, outliers should, under this type of transformation, appear as shown in Figure 3. Empirical evidence of these phenomena, namely reversibility, regime shifts and outliers, will be provided in Section 4.

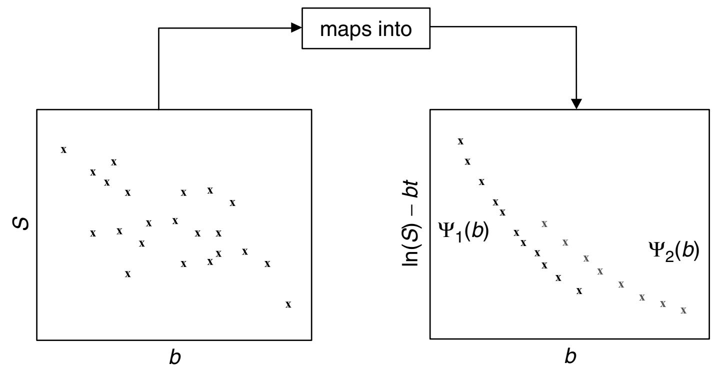


Figure 2: Schematic of how a regime shift manifests itself under the suggested coordinate transformation. A mapping of S versus b into $\ln S - bt$ versus b leads to distinctive characteristic functions, depicted here by $\Psi_1(b)$ and $\Psi_2(b)$, each belonging to a separate regime

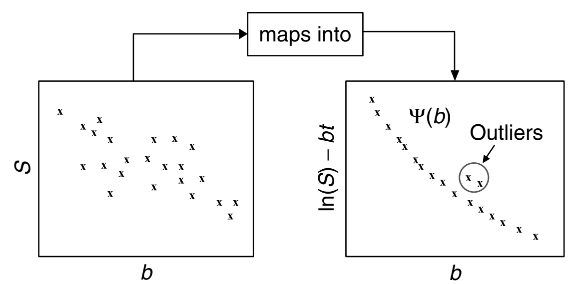


Figure 3: Schematic of how outliers become visible under the suggested coordinate transformation. A mapping of S versus b into $\ln S - bt$ versus b should clearly separate outliers from the function, $\Psi(b)$

3.2.2 The interest rate As mentioned at the end of Section 3.1, the issue of the interest rate is an important one. Putting it more precisely, what should one use for b in Equations 3.6 and 3.8a,b in order to test their validity?

Obviously, several choices exist. These include all the different yields associated with the different, available bond maturities, thus adding to the subjectivity. But, nevertheless, an attempt is made later to settle this point.

Upon following the steps that led to the coordinate transformations in Equations 3.6 and 3.8a,b, it is noted that (bond) maturity or tenor does not enter into the picture. Furthermore, in the context of the reversibility property discussed earlier, it should also not matter which interest rate is used. In other words, using b as the yield of any bond maturity, be it 2 years or 7 years or 30 years, etc., should be acceptable, but only if one moves along a characteristic line, i.e. $\Psi(b)$, $\Phi(b)$, and $\Xi(b)$, which belongs to a certain structural regime. The invariance towards maturity should not be expected to hold across regime shifts and/or to outliers.

4 Evidence of reversibility, outliers and structural shifts

If the hypotheses put forward above were to be proven valid, then upon plotting $\ln X - bt$ against b, where X could signify S, V_G or V_E, one should expect to obtain a single curve, or, more generally, a series of curves, each pertaining to some particular structural regime in the market and/or the economy. Furthermore, it was argued that b could represent the yield associated with any tenor. Examples of each of these, with specific applications to the US, UK and Japan (JP) economies and markets, will be provided in the following sections. Prior to this, however, one must carefully study Table 1, which illustrates how the functions $\Psi(b)$, $\Phi(b)$, and $\Xi(b)$ are calculated.

4.1 Applications to US data

To evaluate the long-run applicability of the model to the US market, refer to Figures 4a,b, where in Figure 4a the S&P price data from 1950 to 2000[13] are plotted both in raw form, as S versus b, and transformed, as $\ln S - bt$ versus b, where b has been chosen to be the 10-year US government bond yield.[14] It is evident here that the raw data, as plotted in Figure 4a, exhibit no regular pattern, whereas the mapped form in Figure 4b definitely displays a convergence that is consistent with theory. A similar conclusion can be derived also from Figures 5a,b, where the aggregated earnings are displayed, both raw and transformed, over the same time period.[15]

Shorter-term, but more detailed, data (quarterly as opposed to annual) for the US, covering from about 1980 to 2004, are presented in Figures 6a–c, where evidence of all the above-mentioned effects, namely convergence, regime shifts and outliers, are clearly depicted. In all instances that follow from now on, the data come from Datastream, using the codes tabulated in Table 2. Also, unless otherwise specified, b will be given by the 10-year government bond yield.

Figures 6a–c present plots of quarterly numbers pertaining to the S&P500 price, I/B/E/S earnings forecast and US GDP, respectively, comparing the raw data against their mapped counterparts. Convergence is noticeable in all cases, although the support is more compelling in the earnings and GDP plots shown in Figures 6b and 6c.

Figure 6a, which pertains to the price index, demonstrates how an outlier, which could otherwise remain hidden in the raw data, stands out in the mapped plane. The outlier highlighted here represents the quarter just before the August 1987 crash, when the overpricing in the S&P500 index, which was then also present in many other national and international indices, led subsequently to the crash.

Figures 6b and 6c, on the other hand, depict structural breaks and regime shifts in the aggregated earnings and GDP. In the interest of objectivity, however, as well as owing to the primary focus of this work, which is to introduce the capabilities of the model rather than guess the causes that could have led to these shifts, there will be no further speculation here. An economist is, perhaps, better suited to undertake this task, by observing the timing of these breaks and connecting them to fundamental (economic and/or market) changes that might have occurred then.

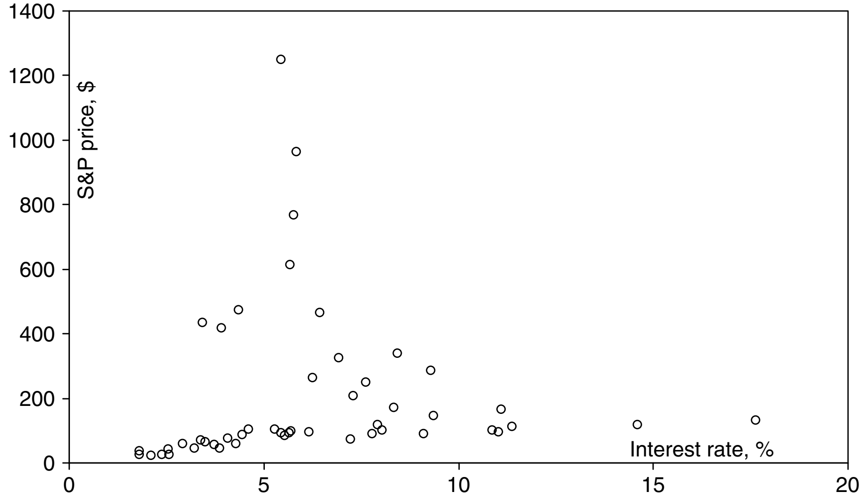
4.2 Applications to UK and JP data

The UK data, concentrating on the FTSE100 price index, aggregated I/B/E/S earnings forecasts and GDP, are presented in Figures 7a–c, respectively. Once again, similar to the US case in

TABLE 1: CALCULATION OF THE FUNCTIONS $\Psi(b)$, $\Phi(b)$ AND $\Xi(b)$ FROM DOWNLOADED US DATA OF THE S&P500 PRICE INDEX, S, 10-YEAR YIELD GOVERNMENT BOND YIELD, b, NOMINAL GDP LEVEL, G, AND THE I/B/E/S AGGREGATE EARNINGS FOR THE INDEX, E_f. DATE OF DOWNLOAD IS FEBRUARY 12, 2004. THE SAMPLE CALCULATIONS IN NOTES 6–10 UNDERNEATH THE TABLE ARE BASED ON Q1 2001

Date	S (1)	b (2)	G (3)	E_f (4)	t (5)	$V_G(b)$ (6)	$V_E(b)$ (7)	$\Psi(b)$ (8)	$\Phi(b)$ (9)	$\Xi(b)$ (10)
Q1 00	1346.09	6.588	9629.4	61.134	19	152167.6	928.0	5.9532	5.2810	5.5813
Q2 00	1437.2	6.556	9822.8	63.983	19.25	153877.4	975.9	6.0084	5.2819	5.6214
Q3 00	1491.72	5.905	9862.1	64.088	19.5	170977.1	1085.3	6.1562	5.4978	5.8382
Q4 00	1367.72	5.696	9953.6	63.668	19.75	178965.9	1117.8	6.0959	5.5700	5.8941
Q1 01	**1301.53**	**5.223**	10024.8	**60.731**	**20**	**197765.7**	**1162.8**	**6.1267**	**5.7502**	**6.0140**
Q2 01	1291.96	5.398	10088.2	58.173	20.25	193188.2	1077.7	6.0708	5.6783	5.8895
Q3 01	1161.97	4.924	10096.2	56.569	20.5	214094.2	1148.8	6.0485	5.8648	6.0371
Q4 01	1138.65	4.888	10193.9	52.872	20.75	217342.5	1081.7	6.0233	5.8750	5.9720
Q1 02	1104.18	4.919	**10329.3**	54.423	21	218251.7	1106.4	5.9739	5.8604	5.9759
Q2 02	1106.59	5.239	10428.3	56.385	21.25	207037.6	1076.3	5.8958	5.7274	5.8680
Q3 02	928.77	4.29	10542	56.342	21.5	258904.4	1313.3	5.9115	6.1419	6.2580
Q4 02	900.36	4.006	10623.7	55.302	21.75	280736.4	1380.5	5.9315	6.2739	6.3589
Q1 03	851.17	3.93	10735.8	55.078	22	#N/A	1401.5	5.8820	#N/A	6.3807
Q2 03	944.3	3.447	10846.7	56.457	22.25		1637.9	6.0835		6.6342
Q3 03	999.74	4.401	11107	58.581	22.5		1331.1	5.9173		6.2035
Q4 03	1034.15	4.146	11246.3	60.08	22.75		1449.1	5.9981		6.3355
Q1 04	1151.82	4.031	#N/A	62.225	23		1543.7	6.1220		6.4148

(1) S&P500 price index
(2) US 10-year government bond yield.
(3) Nominal level of US GDP.
(4) I/B/E/S earnings forecast.
(5) Reference time, with Q1 00 being taken as 19. The initial value has no impact whatsoever on the final results. Quarterly movements are in steps of 0.25.
(6) Calculation of V_G based on Equation 3.7a, i.e. **19 7765.7** = **10 329.3***100/**5.223**. Note that the GDP forecast at time t, $G_f(t)$, is taken as the value of G a year later, i.e. $G_f(t=20) = G(t=21)$.
(7) Calculation of V_E based on Equation 3.7b, i.e. **1162.8** = **60.731***100/**5.223**. Note that the earnings forecast at time t, $E_f(t)$, is taken as the I/B/E/S forecast at time $t = 2$.
(8) Transformed price, based on Equation 3.6. Computed as **6.1267** = ln(**1301.53**) − **5.223*20**/100.
(9) Transformed GDP, based on Equation 3.8a. This is computed here as **5.7502** = ln(**197 765.7**) − **5.223*20**/100 − 5.4. The factor of 5.4 has been subtracted at the end to adjust the level of $\Xi(b)$ to about $\Phi(b)$.
(10) Transformed I/B/E/S earnings forecast, based on Equation 3.8b. Computed as **6.0140** = ln(**1162.8**) − **5.223*20**/100.



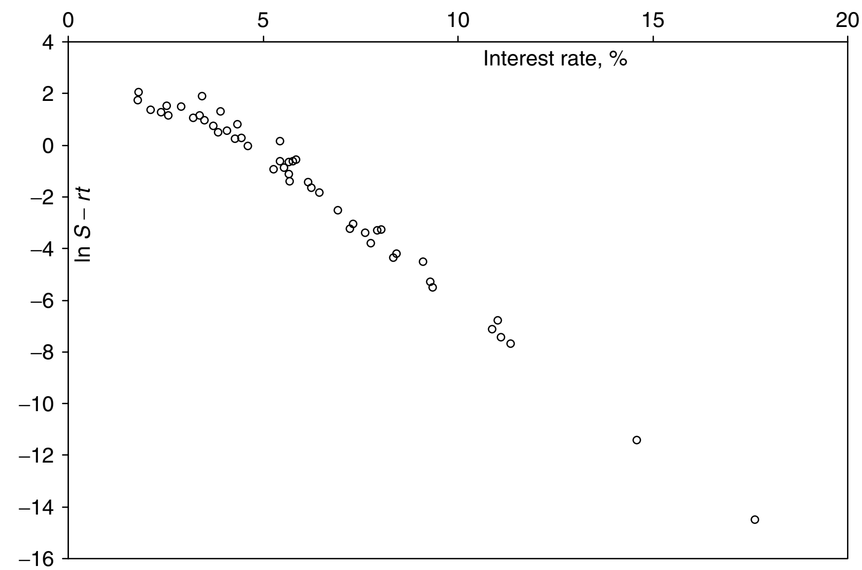


Figure 4: Raw and transformed data, respectively, of the S&P price versus the interest rate from 1950 to about 2000. Transformation of the price data is carried out according to Equation 3.6

Figure 6a, the FTSE100 price index, when mapped, depicts outliers that coincide exactly with time periods immediately prior to the August 1987 crash. In addition, evidence of structural breaks can also be observed in mapped plots of both earnings and GDP.

The JP data, which are included in Figures 8a–c, are substantially different. First, the impact of the transformation on the TOPIX price, as depicted in Figure 6a, is non-existent. Obviously,

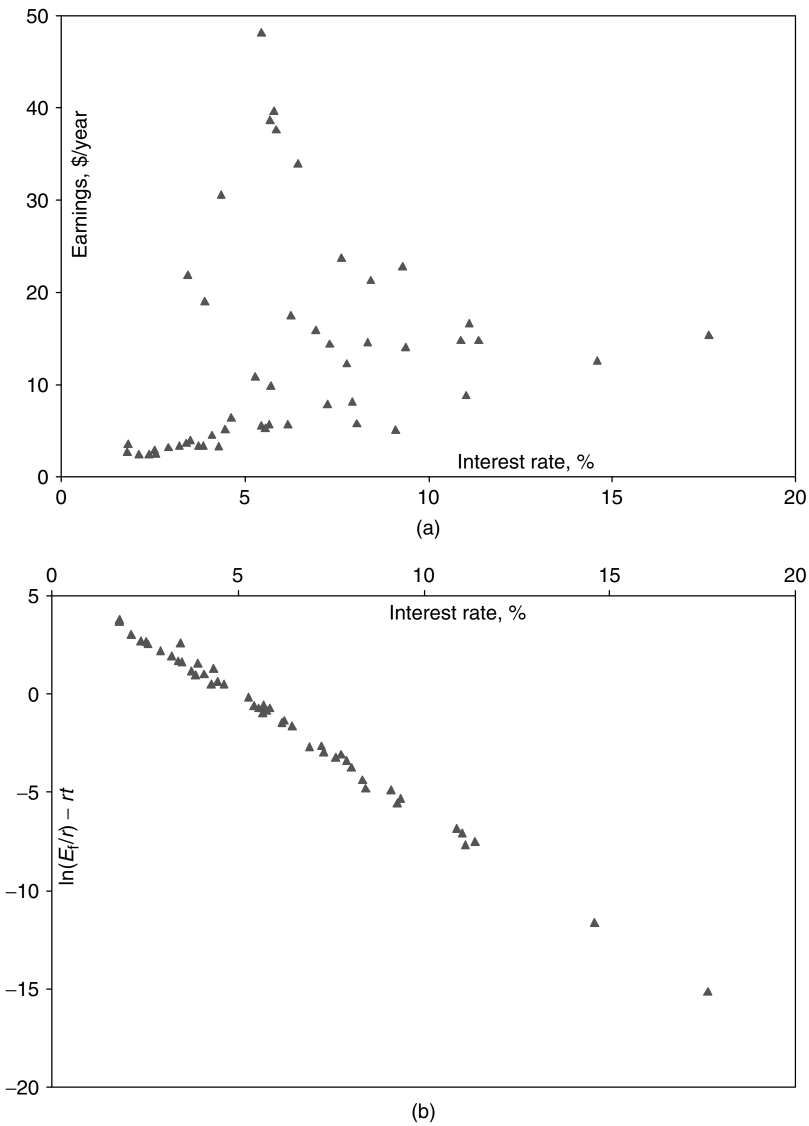


Figure 5: Raw and transformed data, respectively, of the S&P aggregated earnings versus the interest rate from 1950 to about 2000. Transformation of the earnings data is carried out according to Equations 3.7a and 3.8a

the TOPIX does not abide by the same rules that the S&P500 and FTSE100 indices do. As to the reason for this, whether it is a different valuation technique that underlies the TOPIX or a complete detachment between this index and the bond yield (i.e. inapplicability of Equation 3.2 to the TOPIX) is not up for speculation here. What is clear altogether is that this approach does not work for the TOPIX and, hence, cannot be used here.

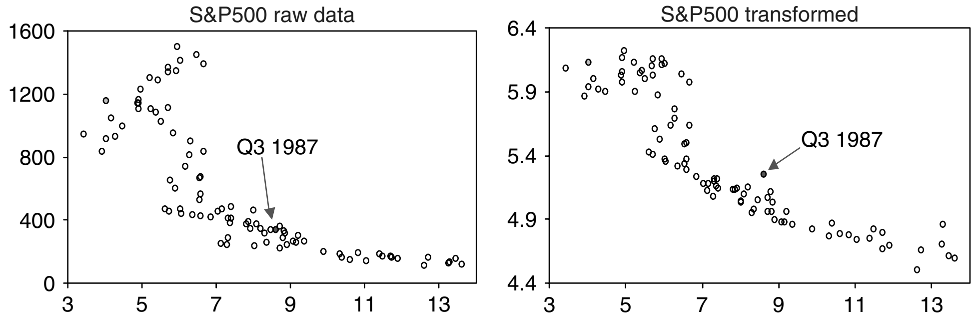


Figure 6a: The S&P500 price index raw data plotted against the 10-year US government bond yield (left) and its transformed counterpart (right). Note the highlighted point representing the quarter prior to the August 1987 crash, where the market was known to be overpriced. Time frame for the plot is Q1 1981 to Q1 2004. The darker point on the top left-hand side is the most current

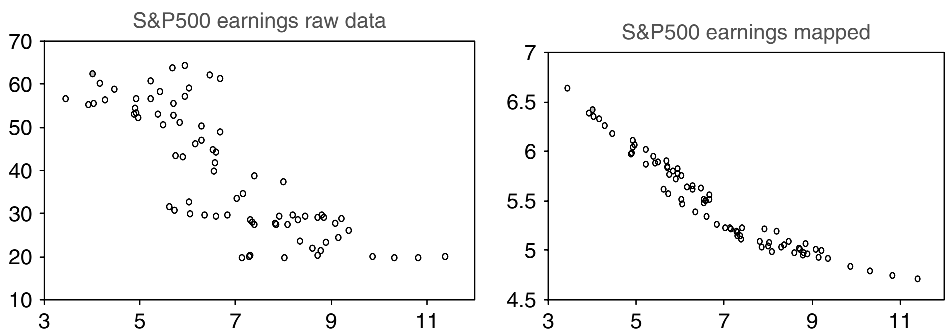


Figure 6b: The S&P500 I/B/E/S earnings forecast raw data plotted against the 10-year US government bond yield (left) and its transformed counterpart (right). Note the existence of a regime shift, similar to that portrayed in Figure 2. Time frame for the plot is Q1 1981 to Q1 2004. The darker point on the top left-hand side corresponds to the latest

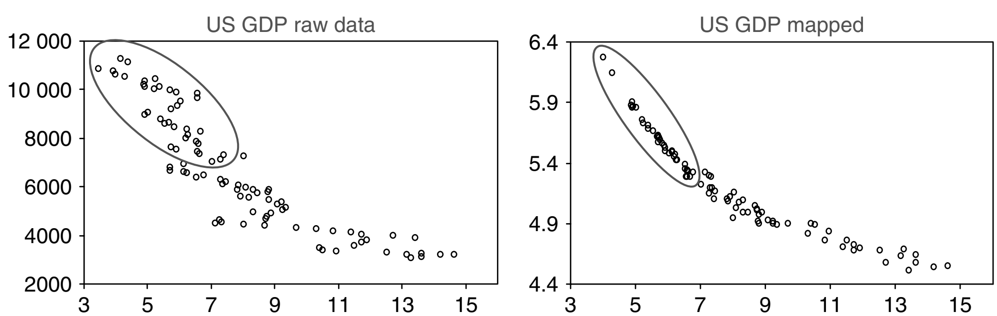


Figure 6c: The US GDP raw data plotted against the 10-year US government bond yield (left) and its transformed counterpart (right). Once again, note the existence of regime shifts. Time frame for the plot is Q1 1981 to Q1 2004. The encircled region covers Q1 2000 to Q4 2003, which, as it appears on the right-hand plot, belongs to a single structural regime and appears to contain no outliers

TABLE 2: DATASTREAM CODES FOR THE QUARTERLY DATA USED IN FIGURES 6 AND THEREAFTER

Country	Parameter	Datastream code
US	S&P500	S&PCOMP
	I/B/E/S earnings forecast	@:USSP500(A12FE)
	US GDP	USGDP . . .B
	30-year US gov. bond yld.	BMUS30Y(RY)
	10-year US gov. bond yld.	BMUS10Y(RY)
	7-year US gov. bond yld.	BMUS07Y(RY)
	5-year US gov. bond yld.	BMUS05Y(RY)
	2-year US gov. bond yld.	BMUS02Y(RY)
UK	FTSE100	FTSE100
	I/B/E/S earnings forecast	@:UKFT100(A12FE)
	UK GDP	UKGDP . . .B
	20-year UK gov. bond yld.	BMUK20Y(RY)
	10-year UK gov. bond yld.	BMUK10Y(RY)
	7-year UK gov. bond yld.	BMUK07Y(RY)
	5-year UK gov. bond yld.	BMUK05Y(RY)
	2-year UK gov. bond yld.	BMUK02Y(RY)
JP	TOPIX	TOKYOSE
	I/B/E/S earnings forecast	@:JPTOPIX(A12FE)
	JP GDP	JPGDP . . .B
	30-year JP gov. bond yld.	BMJP30Y(RY)
	10-year JP gov. bond yld.	BMJP10Y(RY)
	7-year JP gov. bond yld.	BMJP07Y(RY)
	5-year JP gov. bond yld.	BMJP05Y(RY)
	2-year JP gov. bond yld.	BMJP02Y(RY)

In contrast, however, a pattern does emerge when the I/B/E/S earnings forecasts are transformed, as shown in Figure 8b. Here, there is evidence of a structural shift in the earnings, coinciding to around the end of 1994 when the 10-year yield was approximately 4.5%. The JP GDP, on the other hand, which is illustrated in Figure 8c, displays a remarkably tight pattern, showing no signs of any structural change in the economy, at least from Q1 1984 to Q1 2004, the selected range of the data.

In the case of JP, therefore, one could conclude that bond yields (1) are completely detached from the TOPIX price, (2) have an influence on expected earnings and (3) are tightly coupled to the GDP. This, subsequently, could mean that in Japan, the GDP and TOPIX price are not connected to one another, so that any attempt to infer the direction of the TOPIX price, and possibly other Japanese equity indices, from expected movements in either the interest rates and/or the GDP is doomed to fail.

4.3 The impact of bond maturity

Having thus far concentrated only on the 10-year government bond yield, it is time now to question the applicability of the approach to other bond maturities. According to the governing

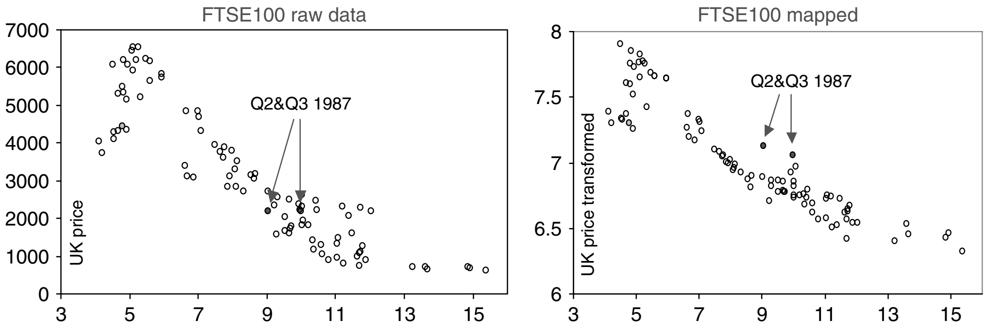


Figure 7a: The FTSE100 price index raw data plotted against the 10-year UK government bond yield (left) and its transformed counterpart (right). Note the highlighted points representing the two quarters prior to the August 1987 crash, where the market was known to be overpriced. Time frame for the plot is Q1 1981 to Q1 2004. The darker point on the top left-hand side corresponds to the latest

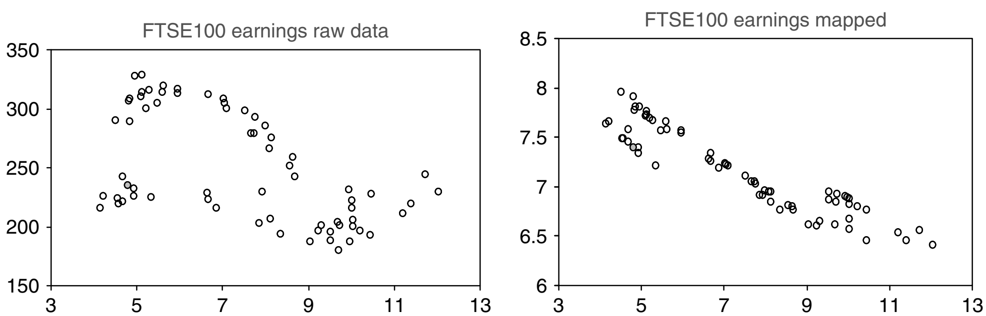


Figure 7b: The FTSE100 I/B/E/S earnings forecast raw data plotted against the 10-year UK government bond yield (left) and its transformed counterpart (right). Note the existence of regime shifts, similar to that portrayed in Figure 2. Time frame for the plot is Q1 1981 to Q1 2004. The darker point on the left-hand side corresponds to the latest

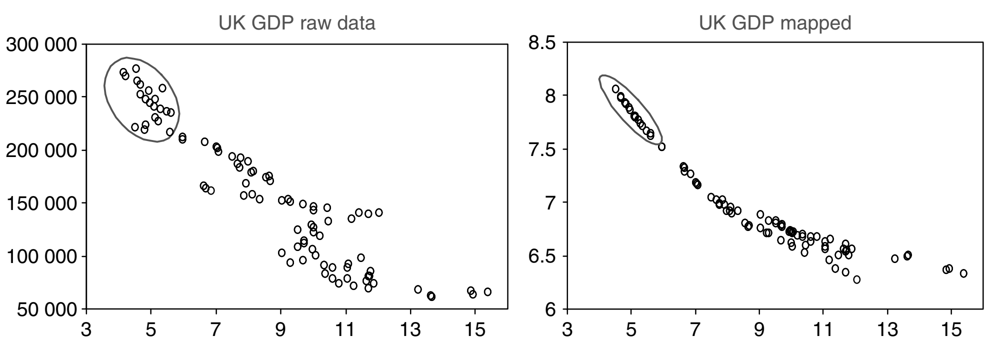


Figure 7c: The UK GDP raw data plotted against the 10-year UK government bond yield (left) and its transformed counterpart (right). Once again, note the existence of regime shifts. Time frame for the plot is Q1 1981 to Q1 2004. The encircled region covers Q1 2000 to Q4 2003, which, as it appears on the right-hand plot, belongs to a single structural regime and contains no outliers

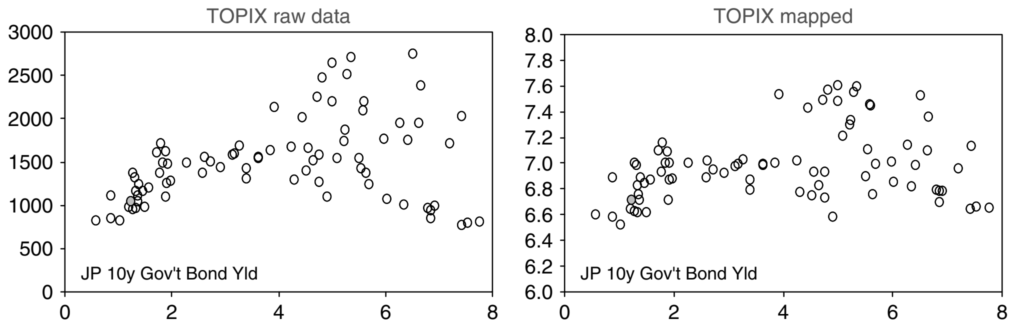


Figure 8a: The TOPIX price index raw data plotted against the 10-year JP government bond yield (left) and its transformed counterpart (right). Note the absence of any convergence in the mapped frame of reference. Time frame for the plot is Q1 1984 to Q1 2004. The darker point on the left-hand side corresponds to the latest

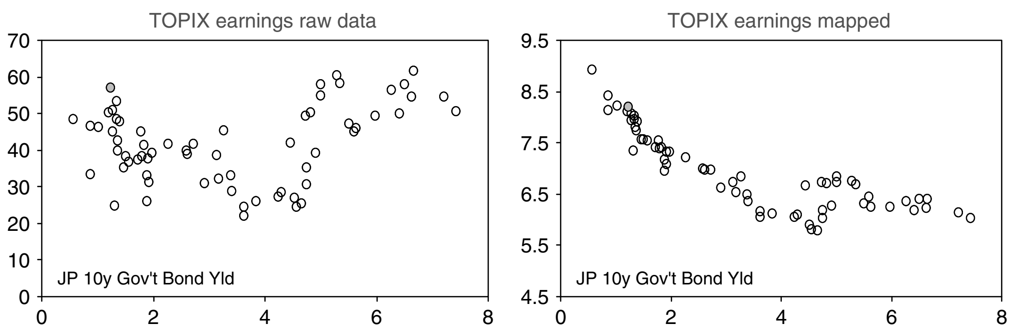


Figure 8b: The TOPIX I/B/E/S earnings forecast raw data plotted against the 10-year JP government bond yield (left) and its transformed counterpart (right). Note the existence of a regime shift, similar to that portrayed in Figure 2. Time frame for the plot is Q1 1984 to Q1 2004. The darker point on the top left-hand side corresponds to the latest

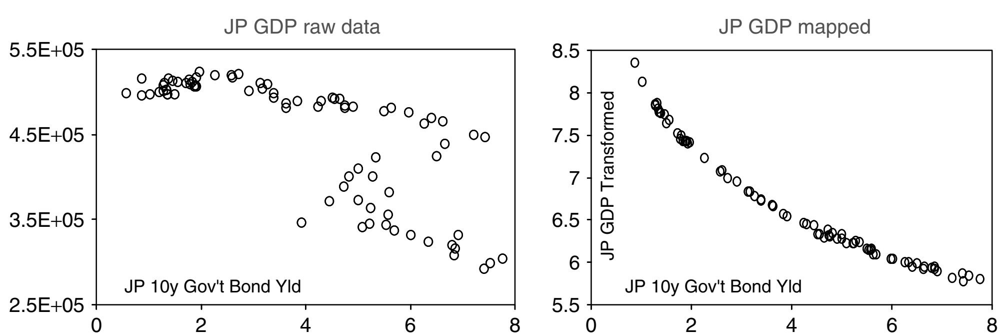


Figure 8c: The JP GDP raw data plotted against the 10-year JP government bond yield (left) and its transformed counterpart (right). Note the absence of regime shifts in the transformed plane, which covers the period Q1 1984 to Q4 2003

equations 3.6–3.8, bond maturity, T, plays *no* role in the model. Therefore, going back to Section 3.2.2, this means that, in the absence of outliers and structural shifts, the characteristic line of convergence in the mapped frame of reference should remain insensitive to the different maturities. More simply stated, all points that result from applying the coordinate transformation using yields from different bond maturities should, under the above conditions, fall exactly on the same line, regardless of maturity.

The validity of the above may now be examined, again visually, by producing plots similar to Figures 6–8. In doing so, care must be taken to select regions where structural shifts and outliers are absent, of which the area encircled in Figure 6c is one. This region contains the time frame Q1 2000 to Q4 2003 for the US GDP. Bearing in mind that the graph was constructed using the 10-year US government bond yield, we now ask what happens if different maturities were also included in the same plot.

The impact of bond maturity on, or rather the absence of its effect in, the present model is clearly demonstrated in Figures 9a–c, which enlarge the areas highlighted in Figures 6c, 7c and 8c, for the US, UK and JP,[16] respectively. In each of these figures, 9a–c, different government bond tenors—namely the 2, 5, 7, 10 and 30 years (20 instead of 30 years in the case of UK)—were plotted together, with the idea that any observable scatter could be attributed to the differences in maturities. Nevertheless, one obtains in all cases a remarkably tight fit, which provides further testimony to the earlier presumption (see Section 3.2.2) that the underlying curve is invariant to different maturities.

5 Potential applications

Prior to going forward with the development of the relative valuation model, two types of applications are brought to mind, both of which could have possible uses in the field of investment.

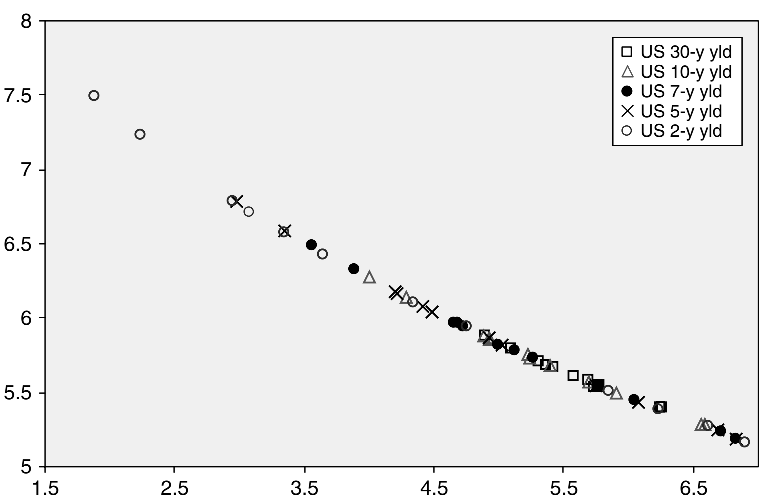


Figure 9a: The transformed US GDP for the area circled in Figure 6c, covering the time frame Q1 2000 to Q4 2003. The plot shows different maturities superimposed on each other. The horizontal and vertical coordinates represent b and $\ln V_G - bt$, respectively

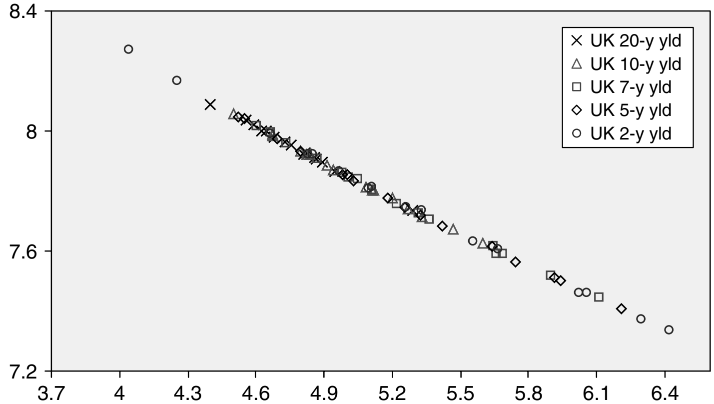


Figure 9b: The transformed UK GDP for the area circled in Figure 7c, covering the time frame Q1 1998 to Q4 2003. The plot shows different maturities superimposed on each other. The horizontal and vertical coordinates represent b and $\ln V_G - bt$, respectively

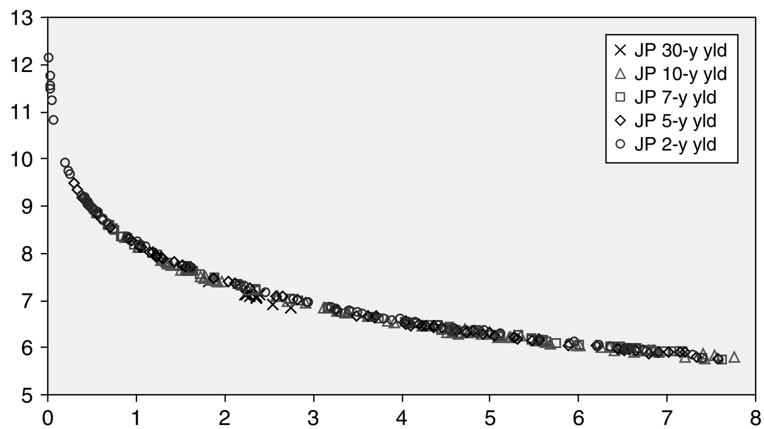


Figure 9c: The transformed JP GDP for all of the area in Figure 8c, covering the time frame Q1 1984 to Q4 2003. The plot shows different maturities superimposed on each other. The horizontal and vertical coordinates represent b and $\ln V_G - bt$, respectively

These applications, which are described next, result from the properties of the curves described in Section 4 and consist of forecasting the GDP and calculating the duration.

5.1 Forecasting the GDP

To illustrate the GDP forecasting capability of the model, one needs to combine Equations 3.7a and 3.8a, replace b^* by b and arrange the result as:

$$G_f(t) = \exp[\Phi(b) + bt + \ln b] \tag{5.1}$$

Recognising that $G_f(t)$ is the GDP expectation, Equation 5.1 then allows one to recover the expected GDP, one year from today, given today's yield, b, as well as the empirically determined function, $\Phi(b)$, which is extractable from plots similar to Figures 9a–c. The assumptions underlying this method are that (1) today's bond yields have the expected GDP priced in them and (2) between now and one year ahead from now, no structural shifts will occur, so that the function $\Phi(b)$ retains its shape over the time period between now and then.

Let us now apply Equation 5.1 to the three cases of interest here, namely the US, UK and JP. Focusing initially on the US, it is observed that a fourth-order polynomial curve runs satisfactorily through all the points in Figure 9a, comprising the yields associated with the different tenors. This curve, therefore, provides an empirical relation for $\Phi(b)$ with an R^2 of 0.99975. The tightness of the fit is noteworthy in Figure 10a, where the polynomial expression is also included.

What follows now is a step-by-step demonstration of how a forecast for the US GDP, let's say of Q1 2005,[17] could be obtained using Equation 5.1. (1) Compute from Table 1 the mapping of GDP to $\Phi(b)$. This, when plotted against b, leads to Figure 10a. (2) A curve fit, similar to the one in that figure, could then be obtained to represent the behaviour of $\Phi(b)$ with respect to b. In this case, a fourth-order polynomial was sufficient to achieve a very tight fit. (3) Return to Equation 5.1 and note that the expected GDP for Q1 2005—i.e. $G_f(t = \text{Q1 2004})$—may now be calculated by substituting the values of b, $\Phi(b)$ and the quantity bt, where, in correspondence to Q1 2004, b and t are 4.031% and 23, respectively (see Table 1).

Repeating the above procedure for the different bond maturities leads to Figure 10b, as well as Table 3, where different estimates of the Q1 2005 expected GDP have been obtained. These fall between 11 350 and 11 906 (in appropriate units), which correspond to the stretch of tenors between 2 and 30 years, respectively. A simple average finally provides an overall estimate of 11 619 for the Q1 2005 US GDP. Note that, since this value is based on yields that are market driven and which tend to vary rather gently on a day-to-day basis, the estimate for one-year-ahead GDP should also behave similarly.

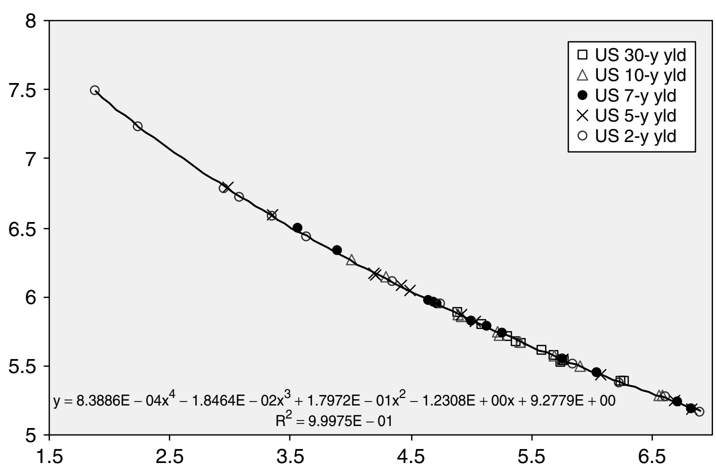


Figure 10a: Same as Figure 9a, but with a fourth-order polynomial curve fit passing through the yields belonging to the different maturities indicated in the legend. The extremely tight fit, as reflected by the high R^2, represents the function $\Phi(b)$

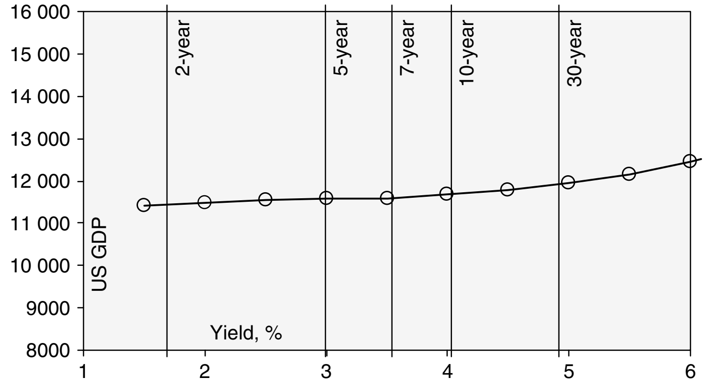


Figure 10b: The expected US GDP for Q1 2005 as a function of the interest rate, as derived from the methodology outlined in Section 5.1. The vertical lines correspond to the different maturity yields as of the time of data download—i.e. February 12, 2004, corresponding to Q1 2004

TABLE 3: EXAMPLE CALCULATION ILLUSTRATING THE GDP FORECASTING PROCEDURE. A VALUE OF 23, CONSISTENT WITH THAT IN TABLE 1, WAS USED HERE FOR t TO SIGNIFY Q1 2004. ALSO, $\Phi(b)$ WAS COMPUTED USING THE POLYNOMIAL FIT IN FIGURE 10a

Maturity, T	Bond yield, b, in %	$bt/100$	$\Phi(b)$	Expected GDP for Q1 05
2 years	1.687	0.38801	7.63116	11 350
5 years	2.998	0.68954	6.77352	11 566
7 years	3.544	0.81512	6.48367	11 601
10 years	4.031	0.92713	6.24891	11 671
30 years	4.911	1.12953	5.86892	11 906
	Average			11 619

To validate these estimates, a back test was performed following the same steps as above. Here, for instance, an estimate for the now historical Q1 2001 GDP level is obtainable from the yields of Q1 2000, as well as upon utilising the same expression for $\Phi(b)$. This back test provides Figures 11a–c, which pertain to the US, UK and JP, respectively. The basis of this is Table 4, which displays the fitted polynomials, as well as the time frames involved, for all three jurisdictions.

5.2 Calculating the duration

In the financial literature, the duration of any parameter, let's say X, is defined as its sensitivity to the interest rate, keeping all else constant. Thus, quantitatively, the duration of X, symbolised here by D_X, is represented by

$$D_X \equiv \left(\frac{\partial \ln X}{\partial b}\right)_t \tag{5.2}$$

Although simplistic in construct, problems abound when trying to calculate D_X in practice. First, since this application involves differentiation, then differentiating any volatile economic or market fundamental, such as the GDP, price, earnings, etc., will lead to even more volatile outcomes. Second, the above definition incorporates a *partial* differentiation with respect to b, which explicitly requires holding the time parameter, t, constant. This is an impossible feat to achieve in practice since expressing Equation 5.2 as the difference, let's say, in GDP level between Q1 2001 and Q1

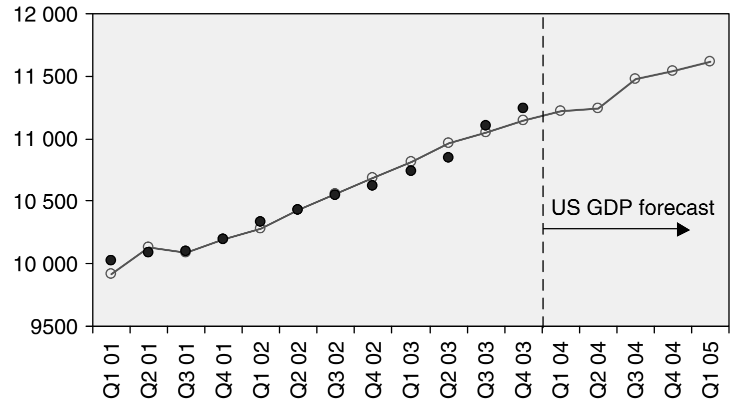


Figure 11a: The US GDP forecast post-Q3 2003 derived by the methodology outlined in Section 5.1. The historical data, which are the solid circles, are also included to demonstrate the close fit between model and data

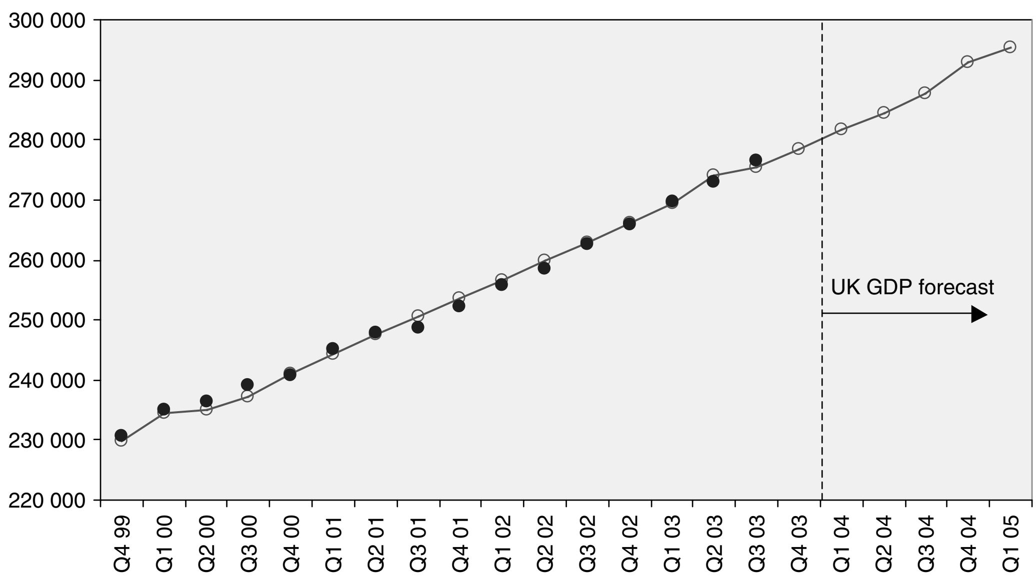


Figure 11b: The UK GDP forecast post-Q3 2003 derived by the methodology outlined in Section 5.1. The historical data, which are the solid circles, are also included to demonstrate the close fit between model and data

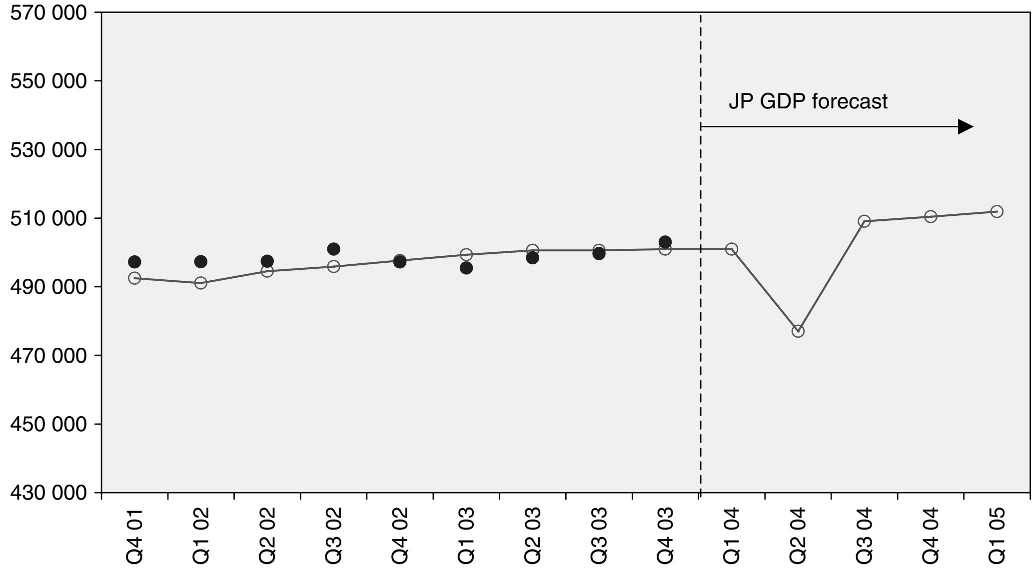


Figure 11c: The JP GDP forecast post-Q3 2003 derived by the methodology outlined in Section 5.1. The historical data, which are the solid circles, are also included to demonstrate the close fit between the model and data

TABLE 4: TIME FRAMES, MATURITIES, POLYNOMIAL FITS AND THE R^2 VALUES UNDERLYING THE CURVES IN FIGURES 9a–c

Market	Time frame of data	Maturities used	4th order polynomial curve fit	R squared
US	Q1 00 - Q1 04	2, 5, 7, 10, 30	8.3886E − 04*b ∧ 4 − 1.8464E − 02*b ∧ 3 + 1.7972E − 01*b ∧ 2 − 1.2308*b + 9.2779	99.975%
UK	Q4 98 - Q1 04	2, 5, 7, 10, 20	−3.730118E − 03*b ∧ 3 + 8.758313E − 02*b ∧ 2 − 0.9.961051*b + 11.10780	99.932%
JP	Q3 00 - Q1 04	5, 7, 10, 30	0.1246*b ∧ 4 − 0.9396*b ∧ 3 + 2.7093*b ∧ 2 − 4.2651*b + 10.509	99.960%

2000 divided by the yield b, will, implicitly, also involve a change in the time parameter. Thus, there is no way in practice that the above expression could be worked out.

Therefore, how could one get around this? Assuming for the time being that X is the GDP level, then, obviously, with $\Phi(b)$ being independent of time, the duration of the GDP—i.e. its sensitivity with respect to the interest rate while holding all else constant—could be computed by simply applying the partial differentiation to it. This yields the expression

$$D_{GDP} = \left(\frac{\partial \ln G_f(t)}{\partial b}\right)_t = \Phi'(b) + t + \frac{1}{b} \tag{5.3}$$

which greatly simplifies the calculation of the duration of GDP, among other data of interest. In practice, therefore, if one were to calculate the sensitivity of the GDP in Q1 2005 with respect to b, then it could be achieved from the above using $\Phi(b)$ in Table 4, along with the appropriate value for t, which, for example, is 23 for the US, in accordance to Table 1. This approach is mathematically more sound than the existing ones simply because the time parameter, t, is literally being held constant in the process of calculating duration.

6 The relative valuation of an equity price index

Thus far, the model has been developed and applied to forecasting the GDP and computing duration. What remains now is its implementation to relative valuation. This is simple as it only involves superimposing the three empirically determined functions, $\Psi(b)$, $\Phi(b)$ and $\Xi(b)$, directly on top of one another and looking for regions of deviation. It should be noted that this method incorporates no adjustable parameters, except for a basic and necessary one that is discussed in note 9 under Table 1.[18] To illustrate how the model works, we start with a preliminary description, along with a couple of historical examples, and then proceed with some detailed assessments.

6.1 A long-term historical example

For a preliminary demonstration, refer to Figures 4a and 4b, where, respectively, the historical price and earnings are mapped against the US government 10-year bond yield. A direct superposition of the two plots leads to Figure 12a, part of which has been magnified in Figure 12b.

Without dwelling too much on this, it is worth noting that the two data series, when mapped as $\Psi(b)$ and $\Xi(b)$ and superimposed, do fall on top of one another over most of the time covered, thus confirming that, with the exception of the period between 1950 and 1960, price and earnings are reasonably valued relative to each other. This chart, nevertheless, is based on annual data and, hence, does not capture the details that are to follow shortly.[19] Before going into these, however, it is worth alluding to an issue that comes up often in related literature—namely, Irving Fisher's assertion that the stock market was not overvalued just before its crash in 1929. An examination of this is carried out in the next section.

6.2 Irving Fisher and the 1929 stock market crash

Let us now apply the model to provide an answer to a long-debated issue, which is whether Fisher was right in his claim that the stock market was *not* overvalued before its dramatic crash in 1929, around the time when the great depression began. This issue seems to be a popular one, as countless papers have been written on it, each attempting to offer an explanation (see, for example, McGrattan and Prescott (2003) and references therein). We shall also try to provide an answer here, albeit strictly in the context of the present model.

Refer to Figure 13, which portrays a superposition of the three functions, $\Psi(b)$, $\Xi(b)$ and $\Phi(b)$, on each other over the time period 1928–1940. Any deviation observed in this mapped plane should, therefore, reflect the degree of relative valuation between the three fundamentals—being price, earnings and GDP.

First, note that from 1928 to 1931, all three fundamentals lie, more or less, near each other, signifying relative fair valuation. The significant deviation, which can be seen as a drop in the

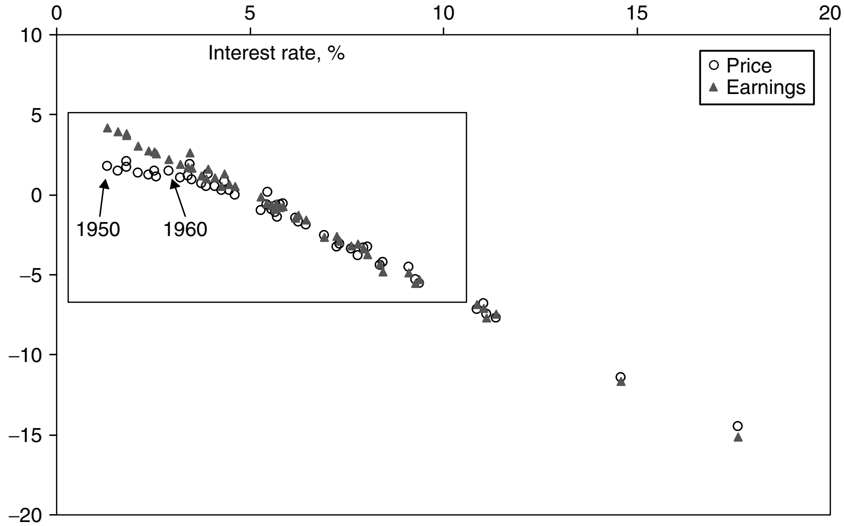


Figure 12a: Figures 4b and 5b superimposed, portraying the notion of relative valuation in the context of this work. Note that the points lie, more or less, on top of one another except for the time frame between 1950 and 1960, during which the convergence was in the process of happening

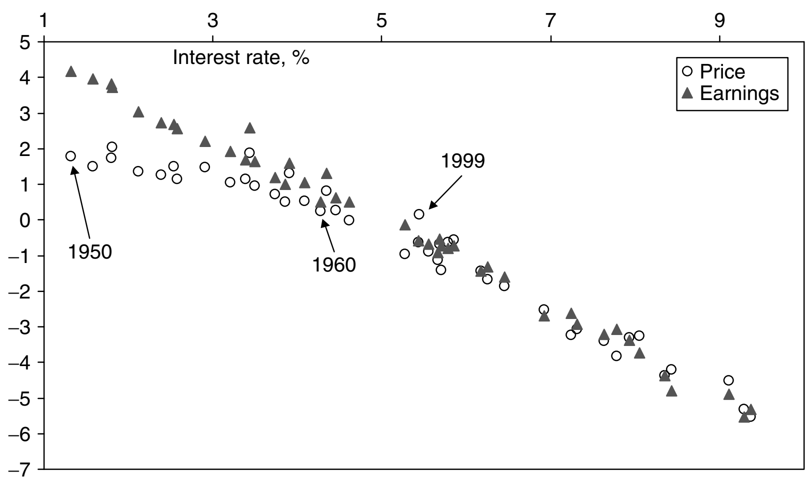


Figure 12b: Magnification of the boxed data in Figure 12a, illustrating the convergence of the two characteristic functions, i.e. $\Psi(b)$ and $\Xi(b)$, at around 1960

mapped price relative to the others, begins at around 1932 and becomes dramatic afterwards. Nevertheless, the mapped earnings and GDP remain reasonably close to one another throughout the whole time period. This, according to the model, means that, just before its plunge, the price was *not* overvalued[20] in relation to the earnings and GDP, but, nevertheless, it did become severely undervalued afterwards. Moreover, the observation that the earnings and GDP remained close to each other during the whole period simply implies that the former reflected the latter fairly well throughout the recession. With this in place, we can go now to the next section and discuss relative valuation in a more current time frame.

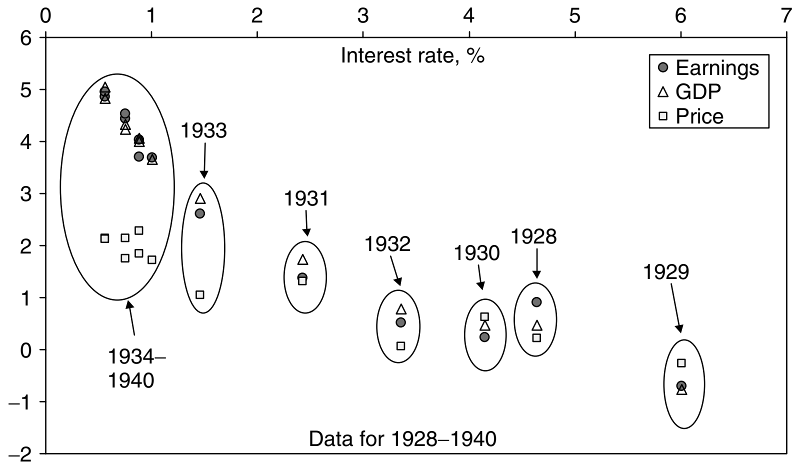


Figure 13a: Superposition of the three functions, $\Psi(b)$, $\Xi(b)$ and $\Phi(b)$, on each other using data covering the period 1928–1940. See Section 6.2 for explanation

6.3 Detailed examination

This section presents a closer look at the more recent time period, whereby the quarterly data displayed in Figures 6–8 are superimposed to exhibit signs of over- and/or undervaluation relative to each other. This is carried out thoroughly for the US and UK, but less so for JP since the TOPIX data, when mapped, lead to inconclusive results (see Figure 8a).

6.3.1 Relative valuation in the US data A relative valuation of the S&P500 price with respect to earnings is illustrated in Figure 14a, revealing the regimes of severe over-undervaluation relative to each other. In this figure, the outlier corresponding to the quarter before the Q3 1987 market crash is highlighted, as well as the time periods of the 1990s tech bubble, the Asian crisis and the post-2001 stock market decline.

Interesting, also, is the close-up view in Figure 14b, focusing on the time frame Q1 1999 to the present, being Q1 2004, and outlining the time-wise progression of the price and yield. This figure essentially displays the dynamics of the price movement, which started initially as overvalued relative to earnings, but eventually crossed the curve at around Q4 2001 to become undervalued, again relative to earnings. In the interest of space, no more will be said here, as the figure is self-explanatory.

Figure 14c displays a superposition of Figures 6a and 6c, relating the behaviours of the S&P500 price and the US GDP. Once again, the 1990s bubble period, as well as the post-2001 market crash, are clearly visible in the shape of deviations of the mapped price, $\Psi(b)$, from the mapped GDP, $\Phi(b)$. Finally, for the US, Figure 14d portrays the mapped S&P500 earnings, $\Xi(b)$, relative to the US GDP. Here, the period coinciding with the 1990s equity bubble is portrayed by a structural regime shift in the shape of a series of earnings data points that fall parallel to, but slightly above, the mapped GDP. Interestingly, however, the post-2001 decline in the market price, which is clearly apparent in Figure 14a, is not reflected at all by the earnings. This supports

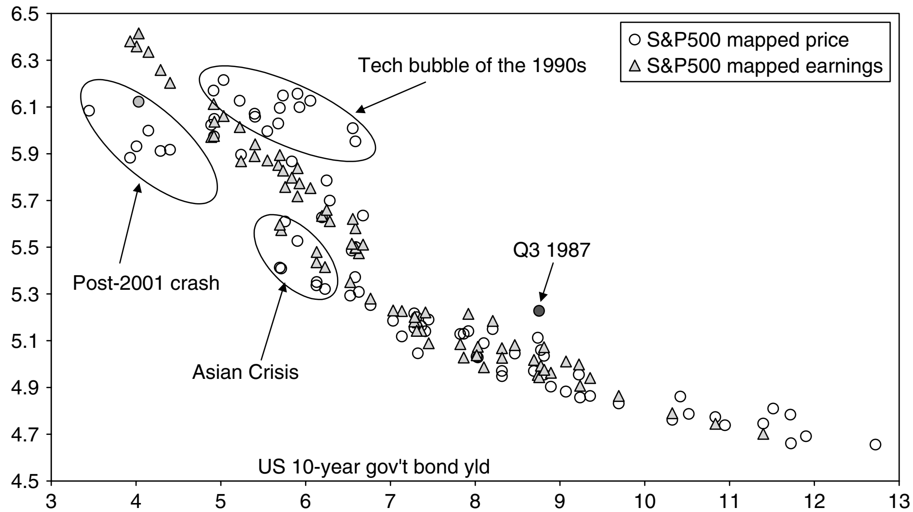
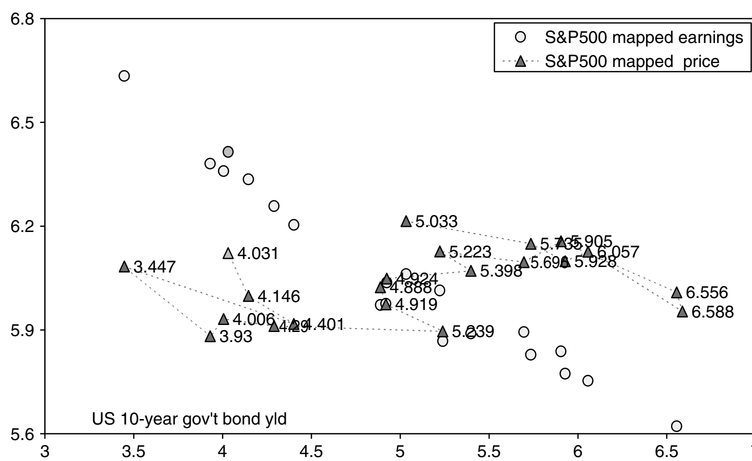


Figure 14a: Relative valuation of the S&P500 price and earnings via superposition of Figures 6a and 6b. Regions of gross deviation are circled. The 1990s tech bubble portrays overvaluation of the stocks relative to earnings and the post-2001 crash shows undervaluation of the former relative to the latter



Q1 99	1237.28	5.033
Q2 99	1333.32	5.735
Q3 99	1332.84	5.928
Q4 99	1424.94	6.057
Q1 00	1346.09	6.588
Q2 00	1437.2	6.556
Q3 00	1491.72	5.905
Q4 00	1367.72	5.696
Q1 01	1301.53	5.223
Q2 01	1291.96	5.398
Q3 01	1161.97	4.924
Q4 01	1138.65	4.888
Q1 02	1104.18	4.919
Q2 02	1106.59	5.239
Q3 02	928.77	4.29
Q4 02	900.36	4.006
Q1 03	851.17	3.93
Q2 03	944.3	3.447
Q3 03	999.74	4.401
Q4 03	1034.15	4.146
Q1 04	1151.82	4.031

Figure 14b: Close-up of Figure 14a, covering the period Q1 1999 to Q1 2004 and depicting the movement of the mapped price relative to mapped earnings. The table on the right-hand side lists the quarter, price and 10-year government bond yield in columns 1, 2 and 3, respectively

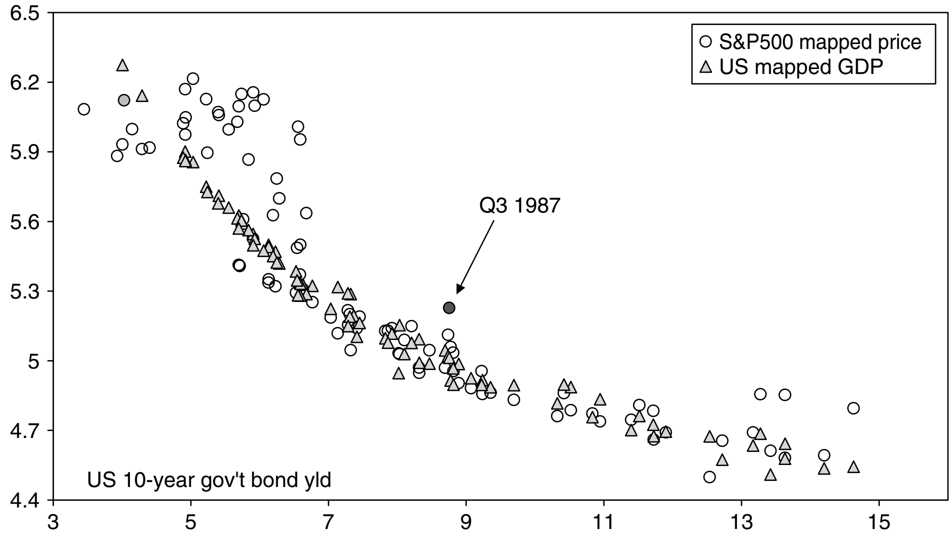


Figure 14c: Relative valuation of the S&P500 price and the US GDP via superposition of Figures 6a and 6c

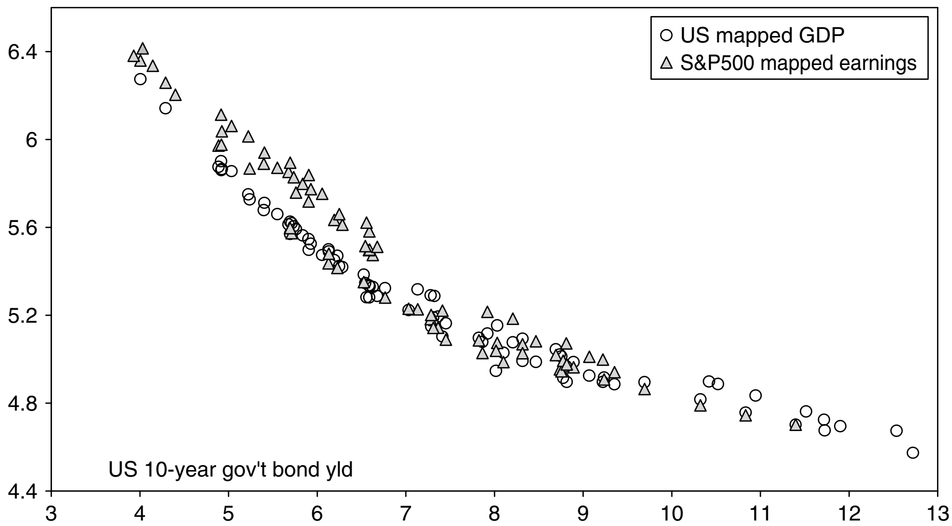


Figure 14d: Relative valuation of the S&P500 earnings and the US GDP via superposition of Figures 6b and 6c

the claim, albeit in retrospect, that the rise in the market's equity price during the 1990s was nothing but a bubble, which ultimately collapsed.

6.3.2 Relative valuation in the UK data Figure 15a, which is a superposition of Figures 7a and 7b, displays the relative behaviour of the FTSE100 price against earnings, both in transformed

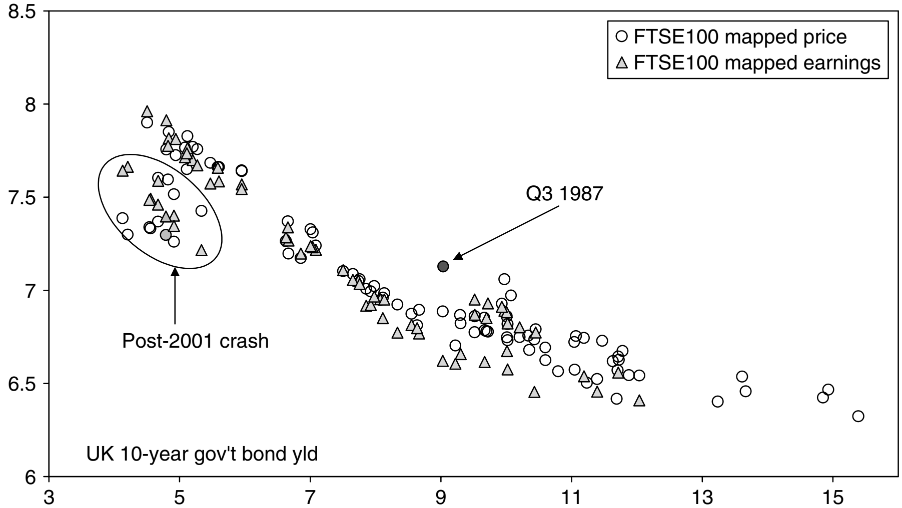
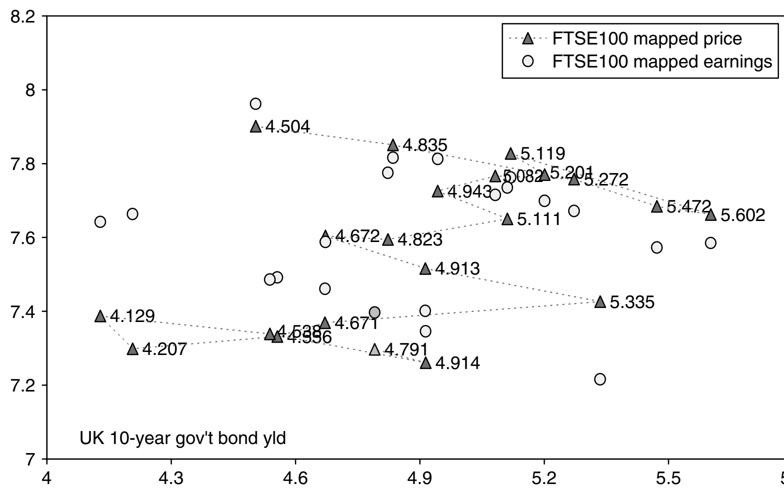


Figure 15a: Relative valuation of the FTSE100 price and earnings via superposition of Figures 7a and 7b. Regions of gross deviation are circled. In contrast to the S&P500 case in Figure 14a, the 1990s tech bubble is completely absent and a structural shift in both price and earnings appears to have occurred post-2001



Q1 99	6074.9	4.504
Q2 99	6206.43	4.835
Q3 99	6201.78	5.201
Q4 99	6550.75	5.119
Q1 00	6164.96	5.602
Q2 00	6232.95	5.472
Q3 00	6543.66	5.272
Q4 00	6440.1	5.082
Q1 01	6088.28	4.943
Q2 01	5914.98	5.111
Q3 01	5342.13	4.823
Q4 01	5290.98	4.672
Q1 02	5154.29	4.913
Q2 02	5217.98	5.335
Q3 02	4329.97	4.671
Q4 02	4115.99	4.556
Q1 03	3729.54	4.207
Q2 03	4049.01	4.129
Q3 03	4272.12	4.538
Q4 03	4354.7	4.914
Q1 04	4442.9	4.791

Figure 15b: Close-up of Figure 15a, covering the period Q1 1999 to Q1 2004 and depicting the movement of the mapped price relative to mapped earnings. The table on the right-hand side lists the quarter, price and 10-year government bond yield in columns 1, 2 and 3, respectively

planes, throughout roughly the last 20 years. The data point pertaining to the quarter prior to the Q3 1987 crash is, once again, highlighted. Here, however, in contrast to the S&P case discussed in Section 6.1.1 and illustrated in Figure 14a, there is no sign, whatsoever, of a price bubble.

In the 1990s, during the peak of the dotcom bubble in the US, the FTSE100 price is observed to follow the earnings consistently. In this case, however, what coincides with the collapse of the

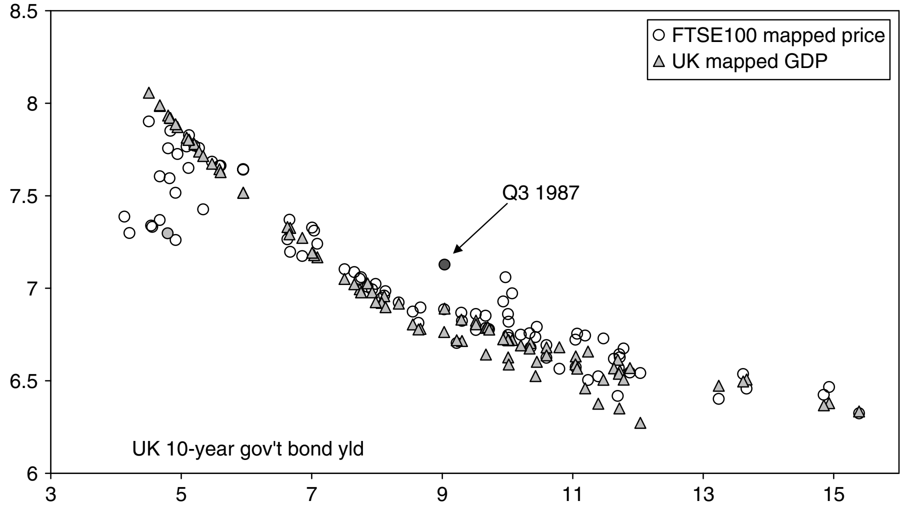


Figure 15c: Relative valuation of the FTSE100 price and the UK GDP via superposition of Figures 7a and 7c

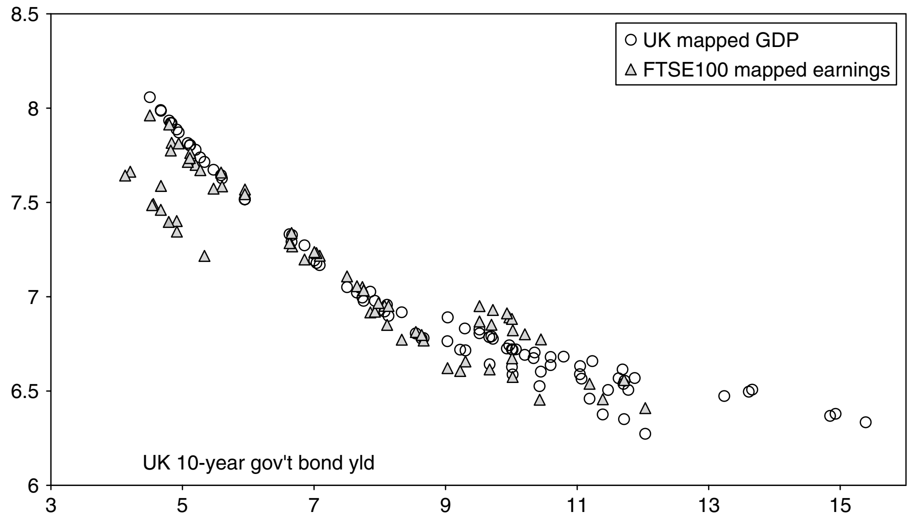


Figure 15d: Relative valuation of the FTSE100 earnings and the UK GDP via superposition of Figures 7b and 7c

price bubble in the S&P is a regime shift in the FTSE100 mapped earnings, which appears also to pull the FTSE100 price with it. This is further confirmed in Figure 15b, where the time-wise movements in earnings and price are depicted in close-up. Again, as in the above and in the interest of remaining objective, we shall not speculate here on the possible reasons for this regime shift (in the behaviour of the earnings and the subsequent fall in the FTSE100 price). Rather, an economist is perhaps better suited to provide an explanation for this.

The lack of a tech bubble, similar to that in the S&P data, in the FTSE price index is again verified in Figure 15c, where the mapped price in Figure 7a is superimposed on the mapped UK GDP in Figure 7b. Moreover, the existence of the regime shift in the FTSE100 earnings, as discussed in the previous paragraph, is found to be quite prominent in Figure 15d, which lays the mapped earnings in Figure 7b directly on top of the mapped UK GDP in Figure 7c.

Altogether, based on the above and without delving into detail, one could deduce that (1) the tech bubble that dominated the S&P500 during the 1990s did not exist in the FTSE100 market and (2) the decline in the FTSE100 price, which coincided with the S&P500 bubble collapse, was initiated by a regime shift in the FTSE earnings. Based on Figure 15d, this regime shift could be 'corrected' by either an increase in the interest rate (to shift the post-2001 earnings line in Figure 15d to the right to match the mapped UK GDP), an increase in earnings (to shift the same line in Figure 15d above to match the UK GDP), or a combination of both. Once the mappings coincide, fair valuation will presumably be achieved between earnings, GDP and price, that is if price will follow earnings.

6.3.3 Relative valuation in the JP data The superimposed JP data are displayed in Figures 16a–c. Figure 16a overlays Figures 8a and 8b, representing the mapped TOPIX price and earnings, respectively. Figure 16b, on the other hand, superimposes the mapped price on the mapped JP GDP in Figure 8c. From the perspective of relative valuation not much can be concluded, as there seems to be no pattern established in the mapped price.

Figure 16c lays the mapped I/B/E/S expected earnings of the TOPIX on top of the mapped JP GDP. There is similarity in the patterns here, although the earnings data converge less tightly and, as already discussed in Section 4.2, they do appear to exhibit some sign of a structural shift,

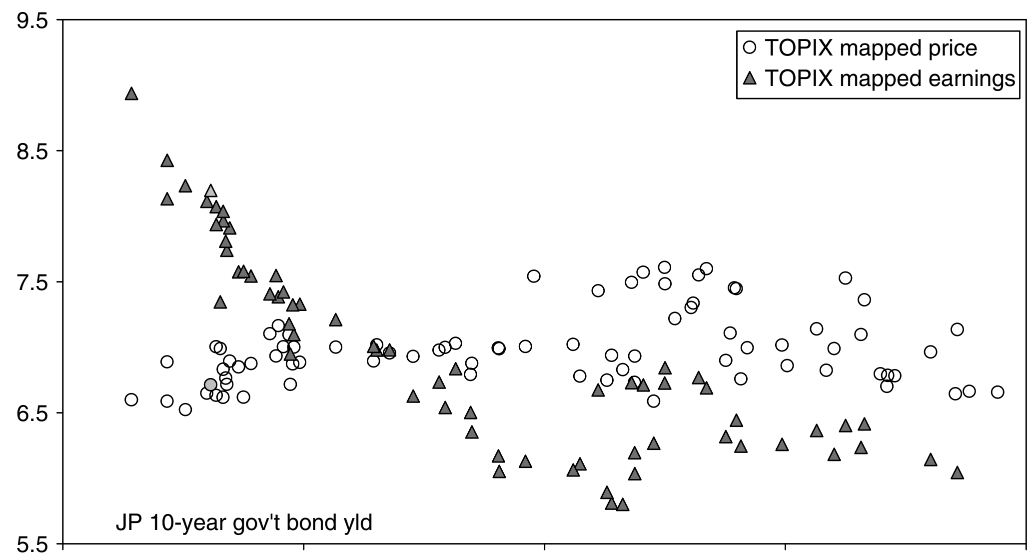


Figure 16a: Superposition of Figures 8a and 8b for the TOPIX mapped price and earnings. The nature of the price prevents any objective assessment of its relative valuation with respect to earnings

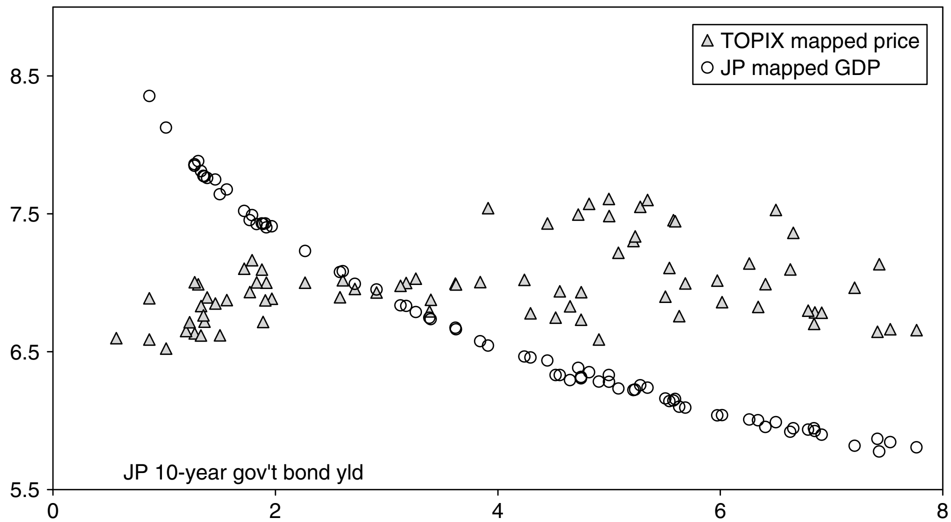


Figure 16b: Superposition of Figures 8a and 8c for the TOPIX mapped price and JP GDP. Once again, as in Figure 16a, the nature of the price prevents any objective assessment of its relative valuation with respect to the GDP

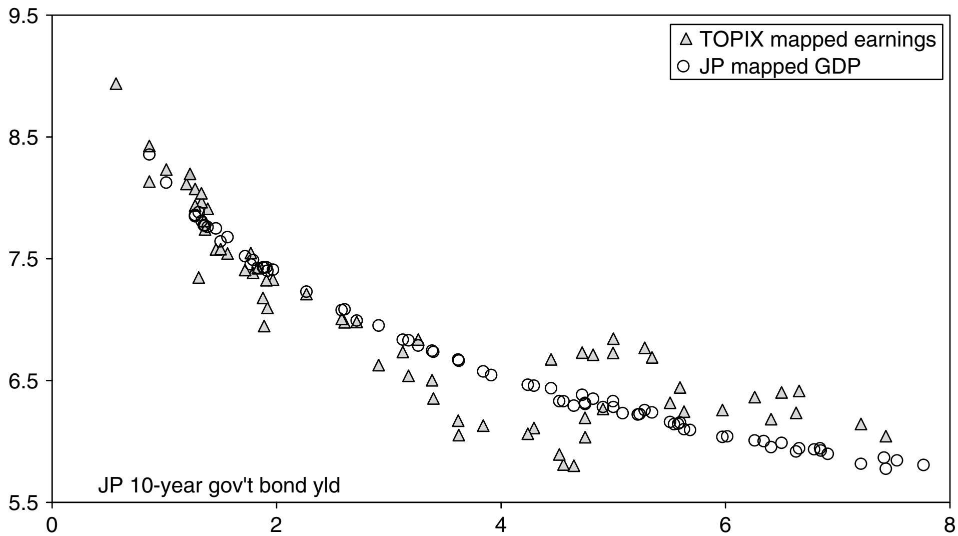


Figure 16c: Superposition of Figures 8b and 8c for the TOPIX mapped earnings and JP GDP

which is absent from the GDP. In terms of relative valuation between the TOPIX earnings and the JP GDP, however, it could be concurred that the two are currently, within the present regime of low interest rates, reasonably close to each other and, hence, the former can be considered to be a fair reflection of the latter.

7 Summary and conclusions

An objective and, hopefully, practical approach to relative valuation of an equity price index has been proposed. The method, which entails a simple mapping, enables one to (1) objectively compare the nominal GDP, corporate earnings and equity index against one another, (2) pinpoint outliers and structural shifts in the data and distinguish between the different regimes, (3) extract an estimate of the GDP forecast for next year, given today's interest rates and (4) obtain a mathematically sound expression for calculating duration. Application of the new method to the US, UK and JP markets and economies led to certain conclusions, some of which are listed below.

1. Fisher's claim that the stock market, just before its dramatic crash in 1929, was not overvalued is supported.
2. A historical, but detailed, assessment of US data, involving the S&P500 price and I/B/E/S earnings forecast, as well as the US GDP, over the last 20 years clearly confirms the existence of the 1990s price bubble in comparison to the earnings and the GDP, and its subsequent collapse in 2001. The collapse brought down the price to fair value relative to both earnings and GDP.
3. An assessment of the UK data, similar to the above, was also undertaken. Here, in contrast to the S&P price data, the results point to the absence of any price bubble in the FTSE100. The subsequent fall in the price, which nonetheless coincided with the collapse of the S&P bubble, occurred as the FTSE100 aggregated earnings underwent a structural shift. A disparate line in Figures 7b, 15a and 15b clearly marks this shift.
4. The situation in JP is markedly different. As depicted in Figures 8a and 16a, b, the mapping transformation has no impact whatsoever on the TOPIX price. In this case, the unmapped price undergoes no change in pattern when subjected to the transformation defined in Equation 3.6. This potentially means that the effect that interest rates or bond yields have on the S&P500 and FTSE100 price indices are totally absent here. As a result, the policy of varying interest rates to manipulate the equity price index does not work in JP under the present circumstances.
   In contrast, the TOPIX earnings and the JP GDP acquire well-defined patterns under the proposed coordinate transformation. A superposition of the two indicates that currently they are both fairly valued relative to each other.

All said, the new model does appear to have some potential as a relative valuation tool and, thereby, might be worth developing further. This could well involve (1) applications to other major equity indices that lie within the same jurisdictions covered here, (2) applications to other jurisdictions and, finally, (3) delving deeper into the other possible uses that were briefly mentioned here—namely, extracting the expected GDP and calculating the duration.

FOOTNOTES & REFERENCES

1. I express these views as an individual, not as representative of companies with which I am connected. E-mail: ruben.cohen@citigroup.com Phone: +44(0)207 986 4645 **Contact address:** Citigroup, London E14 5LB, UK
2. This is also known as the dividend discount model.

3. Note that this is also the return on equity (ROE), which is more an identity rather than a valuation tool.
4. Some might debate here that the DCF or ROE relationship in Equation 2.3 must contain a growth term for the earnings, analogous to the dividend-growth term in Gordon's Growth Model. The argument against including such a term, however, relies on the classical relationship between the plowback ratio and equity growth. The relation, according to literature (see, e.g., Brealey and Myers 1996), as well as intuition, implies that $E_f - \delta_f = \Delta S$, where ΔS is the growth in equity. Dividing both sides of this by the equity, S, leads to an equality between Equations 2.1 and 2.3. This equality first suggests that the total rate of return is the same as the ROE and, second, it reconciles the income statement with the balance sheet. Inclusion of any growth term in Equation 2.3 would, otherwise, produce something inconsistent with the plowback relation provided above.
5. The notion of the risk-free rate is also surrounded by controversy, especially in the empirical literature. Although there is little argument that this number should be based on a government-issued security, questions abound as to what maturity it should take. Another problem, which is more fundamental in nature, addresses the 'riskiness' of the risk-free rate — that is how could government securities be considered risk free when they are, as with any other type of security, volatile and impossible to predict.
6. This, obviously, presents an idealised scenario, but it will be relaxed later as the relative valuation model is developed.
7. Which is especially valid in the absence of volatility.
8. Based on this, therefore, firms pay and/or investors demand dividends because of the uncertainties inherent in the market. Take away these uncertainties — i.e. as per Propositions 1 and 2 — and the dividend yield will disappear altogether from the fundamental relationships, Equations 2.1 and 2.2.
9. Mappings and/or coordinate transformations, whose principal objective is to condense theoretical and empirical data into more manageable formats, have, for nearly a century, played a central role in the field of fluid mechanics. Although a few successful attempts have been made so far to apply this technique to economics (see, for instance, de Jong 1967 and Cohen 1998), as of yet, and as far as we are aware, very few endeavours, if any, have been made to incorporate it into finance.
10. Although materially different in approach from the classical 'dimensional analysis' described in de Jong (1967) and Cohen (1998), among others, the fundamental purpose of the coordinate transformation introduced here remains essentially the same.
11. The DCF model converges with the dividend discount model after 1950 (refer to Cohen 2002).
12. Note the similarity between Equations 3.7b and 2.3, as they are both based on the DCF model.
13. Price and earnings data from Shiller (`http://www.econ.yale.edu/shiller/data/ie_data.htm`). Interest rate data from the Fed website (`http://www.federalreserve.gov/releases/h15/data/m/tcm10y.txt`).
14. As discussed in Section 3.2.2, and as it will also be shown in a later section, the choice of bond maturity does not matter.
15. The earnings data used here are actual, rather than the I/B/E/S forecasts. Therefore, V_E was in this case computed the same way as V_G, where the one-year forward is substituted for

today's forecast of one-year ahead—i.e. $E(t+1)$ used for $E_f(t)$ (refer, for instance, to Table 1 for the method of calculation of $V_G(t)$).

16. Since the JP GDP is all one regime, then Figure 9c contains all the time frame included in Figure 8c.

17. Given that today is Q1 2004.

18. The need for this arises from the scale differential between the GDP and the aggregated index earnings.

19. More detailed, quarterly data will be shown later to clearly capture the 1990s bubble and its collapse.

20. This, therefore, is consistent with Fisher's claim and all the subsequent works that support it.

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