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Self Similarity in Brownian Motion and Other Ergodic Phenomena

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The 1926 Nobel Prize in physics was received by J. Perrin for his precise determination of Avogadro's number, N. His technique was to compare the experimentally measured diffusivity of spherical granules in water with the Stokes-Einstein expression for the diffusion coefficient of spherical particles in an ideal fluid,

$$D = \frac{RT/N}{3\pi\mu d} \qquad (1)$$

where R is the universal gas constant, T is the absolute temperature, μ is the fluid viscosity, and d is the particle diameter.

At that time, Perrin's instruments for determining D consisted essentially of a microscope and a watch. By measuring the square of the particle displacement, $\overline{r^2}$, relative to the origin as a function of time, t, he was able to calculate the diffusion coefficient using the following

$$D = \frac{1}{6}\frac{\overline{r^2}}{t} \qquad (2)$$

where the ⅙ denotes diffusion in the radial direction. Equating expressions 1 and 2 for known values of R, T, μ, d, and D, Perrin was thus successful in obtaining N to a high degree of accuracy.[1,2]

It is well known that the best microscopes and watches available then had resolutions that were orders of magnitude lower than those used today. It is also understood that in consequence of self similarity of Brownian motion Perrin's result would not differ should the experiments be carried on under today's high-quality equipment. The objective here is to demonstrate the case that is universally accepted, but to my knowledge has not yet been explained in simple terms. Moreover, in addition to providing the reader a basic understanding of the nature of self similarity that governs random processes, the calculations presented hereafter may be employed in the treatment of problems with similar character.

Measurement of Brownian Diffusion

Consider the actual path that a particle may undergo Brownian movement as denoted by path I in Figure 1. As a consequence of equipartition of energy, the particle kinetic energy is given by $3kT/2$, and in association with eqs 1 and 2

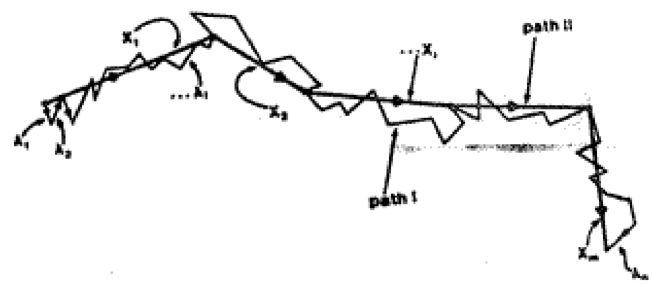

Figure 1. Observing Brownian movement in high (λ_i) and low (X_j) resolutions, paths I and II, respectively.

one can obtain an estimate of the actual mean free path, λ, and the time between collisions, τ, of a spherical particle in a fluid. Hence, for a 0.1-μm particle with density equal to that of water

$$\lambda \cong 5 \times 10^{-4}\,\mu\text{m} \qquad (3)$$

and

$$\tau \cong 3 \times 10^{-9}\,\text{s} \qquad (4)$$

in water at room temperature.

Now assume that two observers are measuring D and that observer 1 is properly equipped to handle the length and time scales given by eqs 3 and 4. His method would then be to use the following

$$D = \frac{1}{6}\frac{\sum_{i=1}^{n}\lambda_i^2}{\sum_{i=1}^{n}\tau_i} \qquad (5)$$

Observer 2, however, being limited in optical and timing resolutions, is capable of resolving length and time scales in the order of X_j and t_j, respectively, as indicated by path II in Figure 1. His measurements would thus lead to

$$D = \frac{1}{6}\frac{\sum_{j=1}^{m}X_j^2}{\sum_{j=1}^{m}t_j} \qquad (6)$$

[1] Perrin, M. J. *Ann. Chem. Phys.* **1909**, Sept. (Translated by F. Soddy, 1910.)
[2] Hersh, R.; Griego, R. J. *Sci. Amer.* **1969**, *220* (March), 66.

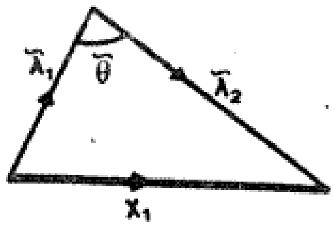

Figure 2. Simplified case of the two paths.

Realizing that the total time,

$$t = \sum_{i=1}^{n} \tau_i = \sum_{j=1}^{m} t_j \tag{7}$$

is the same in both cases, $\sum_{i=1}^{n} \lambda_i^2$ must then be equal to $\sum_{j=1}^{m} X_j^2$ for the calculated values of D to be the same.

Consider the simple case, illustrated in Figure 2, where the particle path consists of two actual paths, $\bar{\lambda}_1$ and $\bar{\lambda}_2$, as measured by observer 1. Being optically limited, observer 2 will see X_1 where

$$X_1^2 = \bar{\lambda}_1^2 + \bar{\lambda}_2^2 - 2\bar{\lambda}_1 \bar{\lambda}_2 \cos \bar{\theta} \tag{8}$$

For maximum accuracy a large number of observations must be made and X_1^2 must be averaged over all the observed values of $\bar{\lambda}_1$, $\bar{\lambda}_2$, and $\bar{\theta}$. Analytically, this is achieved by introducing a probability function[3], $P(\lambda_1,\lambda_2,\theta)$, where

$$P(\lambda_1,\lambda_2,\theta)\, d\lambda_1 d\lambda_2 d\theta$$

gives the number of observations in which the particle is displaced within the limits

$$\lambda_1 \leq \bar{\lambda}_1 \leq \lambda_1 + d\lambda_1$$

$$\lambda_2 \leq \bar{\lambda}_2 \leq \lambda_2 + d\lambda_2$$

and

$$\theta \leq \bar{\theta} \leq \theta + d\theta$$

Thus

$$\overline{X_1^2} = \int_{\theta=0}^{2\pi} \int_{\lambda_2=0}^{\infty} \int_{\lambda_1=0}^{\infty} P(\lambda_1,\lambda_2,\theta)\, [\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos\theta] d\lambda_1 d\lambda_2 d\theta \tag{9}$$

gives the average reading of observer 2 over all possible values of λ_1, λ_2, and θ.

Taking into account the random nature of Brownian motion, λ_1, λ_2, and θ are therefore independent and the probabilities of their occurrence are uncoupled. This implies that

$$P(\lambda_1,\lambda_2,\theta) = P(\lambda_1)\, P(\lambda_2)\, P(\theta) \tag{10}$$

Subsequently, eq 9 becomes

$$\overline{X_1^2} = \int_{\theta=0}^{2\pi} P(\theta) d\theta \int_{\lambda_2=0}^{\infty} P(\lambda_2) d\lambda_2 \int_{\lambda_1=0}^{\infty} \lambda_1^2 P(\lambda_1) d\lambda_1 + \int_{\theta=0}^{2\pi} P(\theta) d\theta \int_{\lambda_1=0}^{\infty} P(\lambda_1) d\lambda_1 \int_{\lambda_2=0}^{\infty} \lambda_2^2 P(\lambda_2) d\lambda_2 - 2 \int_{\lambda_1=0}^{\infty} P(\lambda_1)\lambda_1 d\lambda_1 \int_{\lambda_2=0}^{\infty} P(\lambda_2)\lambda_2 d\lambda_2 \int_{\theta=0}^{2\pi} P(\theta)\cos\theta d\theta \tag{11}$$

Assuming that $P(\lambda)$ approaches zero much faster than $1/\lambda$ (in effect it does due to the Gaussian form of $P(\lambda)$), $\int_{\lambda=0}^{\infty} P(\lambda)\lambda d\lambda$ will therefore approach a finite number. Furthermore, due to randomness $P(\theta)$ is uniform over $0 \leq \theta < 2\pi$, and since by definition

$$\int_{\theta=0}^{2\pi} P(\theta) d\theta = 1 \tag{12}$$

$P(\theta)$ will therefore be equal to $1/2\pi$. This reduces eq 11 to

$$\overline{X_1^2} = \int_{\lambda_1=0}^{\infty} \lambda_1^2 P(\lambda_1) d\lambda_1 + \int_{\lambda_2=0}^{\infty} \lambda_2^2 P(\lambda_2) d\lambda_2 \tag{13}$$

where

$$\int_{\lambda=0}^{\infty} P(\lambda) d\lambda = 1 \tag{14}$$

Equation (13) is similar to that used by Einstein in his statistical definition of D.[3] By carrying on the analysis to include numerous microscopic paths, it follows that

$$\sum_{j=1}^{m} X_j^2 = \sum_{i=1}^{n} \lambda_i^2 = \overline{r^2} \tag{15}$$

indicating that both observers will arrive at the same conclusion that

$$D = \frac{1}{6} \frac{\sum_{i=1}^{n} \lambda_i^2}{\sum_{i=1}^{n} \tau_i} = \frac{1}{6} \frac{\sum_{j=1}^{m} X_j^2}{\sum_{j=1}^{m} t_j} = \frac{1}{6} \frac{\overline{r^2}}{t} \tag{16}$$

independent of instrument resolution.

Concluding Discussion

The results deduced from the preceding section can provide a simple explanation to the nature of many questions encountered by students, engineers, and scientists. For example, how certain low-grade, low-resolution instruments can yield precise and accurate data with a high degree of confidence, and why certain random or ergodic occurrences, such as Brownian motion, exhibit self similarity at any length (and/or time) scale, a behavior that is also evidenced in fractals and percolation.[4,5] Furthermore, the calculations presented above may, to some extent, be related to several common statistical techniques of data reduction, namely the method of least squares, to shed some light upon their apparent effectiveness.

[3] Einstein, A. "Investigations on the Theory of the Brownian Movement"; Furth, R., Ed.; Cowper, A. D., Trans.; Dover: New York, 1956.

[4] Falconer, K. J. "The Geometry of Fractal Sets"; Cambridge University, 1985.

[5] Stauffer, D. "Introduction to Percolation Theory"; Taylor and Francis, 1985.